How long does it take for all users in a social network to choose their communities?

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Abstract

We consider a community formation problem in social networks, where the users are either friends or enemies. The users are partitioned into conflict-free groups (*i.e.*, independent sets in the *conflict graph* $G^- = (V, E)$ that represents the enmities between users). The dynamics goes on as long as there exists any set of at most k users, k being any fixed parameter, that can change their current groups in the partition *simultaneously*, in such a way that they all strictly increase their utilities (number of friends *i.e.*, the cardinality of their respective groups minus one). Previously, the best-known upper-bounds on the maximum time of convergence were $\mathcal{O}(|V|\alpha(G^-))$ for $k \leq 2$ and $\mathcal{O}(|V|^3)$ for $k = 3$, with $\alpha(G^-)$ being the independence number of G^-. Our first contribution in this paper consists in reinterpreting the initial problem as the study of a dominance ordering over the vectors of integer partitions. With this approach, we obtain for $k \leq 2$ the tight upper-bound $\mathcal{O}(|V| \min\{\alpha(G^-), \sqrt{|V|}\})$ and, when G^- is the empty graph, the exact value of order $\frac{(2|V|)^{3/2}}{3}$. The time of convergence, for any fixed $k \geq 4$, was conjectured to be polynomial [7, 14]. In this paper we disprove this. Specifically, we prove that for any $k \geq 4$, the maximum time of convergence is an $\Omega(|V|^{\Theta(\log |V|)})$.

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Foreword: The organizers of a wedding (party) have difficulties in arranging place settings for the guests as there are many incompatibilities among those who do not want to be at the same table as an "enemy" (ex girl (boy) friend, boss or employee, student or supervisor, etc...). The organizers realize that they have no set of 5 pairwise friends and so allow people place themselves. Successively each person joins a table where she has no enemies or starts a new table. At any time a person can move from one table to another table (of course where she has no enemy) if in doing so she increases strictly the number of

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[2] Part of this work has been done during his post-doc at Columbia University in the City of New York.

friends she has at the new table. The process converges relatively fast (linear time). Some time later the organizers of FUN having heard about this scenario decide to use the same process to place the participants in different groups for the social activities of the afternoon. Each participant registers first in her own group. The organizers decide to accelerate the process by authorizing not just one person but any subset of 4 persons to change their mind and leave the group in which they are registered to join another group or create a new group; these persons move only if they desire to do so, that is, they increase strictly the number of friends. Surprisingly the process takes a very long (exponential) time and night arrives before groups are formed. As we will discover, the exponential time derives from the fact that at FUN all the persons are friends and there are no enemies due to the use of moves implying 4 persons. At this point the reader (and the organizers) might ask why we see such a difference in behaviors and how long does it takes for users of a social network to form groups. The answers to these questions and "all you wanted to know but were afraid to ask" will be revealed in this paper.

1 Introduction

Community formation is a fundamental problem in social network analysis. It has already been modeled in several ways, each trying to capture key aspects of the problem. The model studied in this paper has been proposed in [14] in order to reflect the impact of information sharing on the community formation process. Although it is a simplified model, we show that its understanding requires us to solve combinatorial problems that are surprisingly intricate. More precisely, we consider the following dynamics of formation of groups (communities) in social networks. Each group represents a set of users sharing about some information topic. We assume for simplicity that each user shares about a given topic in only one group. Therefore the groups will partition the set of users. We follow the approach of [14]. An important feature is the emphasis on incompatibility between some pairs of users that we will call enemies. Two enemies do not want to share information and so will necessarily belong to different groups. In the general model one consider different degrees of friendship or incompatibilities. Here we will restrict to the case where two users are either friends or enemies – as noted in [14], even a little beyond this case, the problem quickly becomes intractable. As example, if we add a neutral (indifference) relation, there are instances for which there is no stability.

The social network is often modeled by the friendship graph G^+ where the vertices are the users and an edge represents a friendship relation. We will use this graph to present the first notions and examples. However, for the rest of the article and the proofs we will use the complementary graph, that we call the *conflict graph* and denote by G^-; here the vertices represent users and the edges represent the incompatibility relation. We assign each user a *utility* which is the number of friends in the group to which she belongs. Equivalently, the utility is the size of the group minus one, as in a group there is no pair of enemies; in [14] this is modeled by putting the utility as $-\infty$ when there is an enemy in the group.

In the example of Figure 1, the graph depicted is the friendship graph: the edges represent the friendship relation, and if there is no edge it corresponds to a pair of enemies. Figure 1(a) depicts a partition of 12 users composed of 4 non-empty groups each of size 3. The integers on the vertices represent the utilities of the users which are all equal to 2. Figure 1(b) depicts another partition consisting of 5 groups with one group of size 4 (where users have utility 3) and 4 groups of size 2 (where users have utility 1).

In this study we are interested in the dynamics of formation of groups. Another important

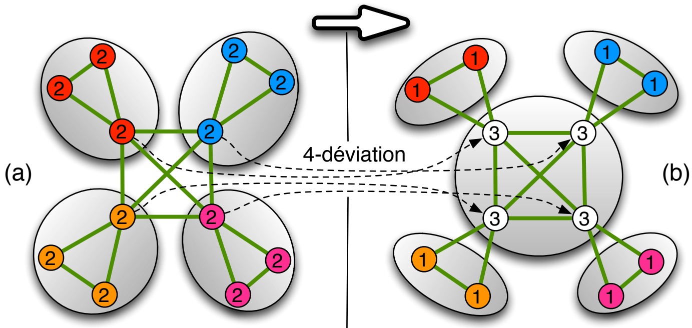

Figure 1 A friendship graph with 12 vertices (users). **(a)** 3-stable partition that is not 4-stable but it is optimal in terms of total utility. **(b)** k-stable partition for any $k \geq 1$ that is not optimal in terms of total utility.

feature of [14], taken into account in the dynamics, is the notion of bounded cooperation between users. More precisely, the dynamics is as follows: initially each user is alone in her own group. In the simplest case, a move consists for a specific user to leave the group to which she belongs to join another group but only if this action increases strictly her utility (acting in a selfish manner); in particular, it implies that a user does not join a group where she has an enemy. In the k-bounded mode of cooperation, a set of at most k-users can leave their respective groups to join another group, again, only if each user increases strictly their utility. If the group they join is empty it corresponds to creating a new group. We call such a move a *k-deviation*. Note that this notion is slightly different from that of $(k+1)$-defection of [14]. We will say that *a partition is k-stable* if there does not exist a k-deviation for this partition.

The partition of Figure 1(a) is k-stable when $k \in \{1, 2, 3\}$. Indeed each user has at least one enemy in each non empty other group and so cannot join another group. Furthermore, when $k \leq 3$, if k users join an empty group their utility will be at most 2 and so will not strictly increase. However, this partition is not 4-stable because there is a 4-deviation: the four central users can join an empty group and so they increase their utilities from 2 to 3. The partition obtained after such a 4-deviation is depicted in Figure 1(b). This partition is k-stable for any $k \geq 1$. Note that the utility of the other users is now 1 (instead of 2). Thus, we deduce that this partition is not optimal in terms of total utility (the total utility has decreased from 24 to 20); but it is now stable under all deviations. This illustrates the fact that users act in a selfish manner as some increase their utility, but on the contrary the global utility decreases. For more information on the suboptimality of k-stable partitions, *i.e.*, bounds on the price of anarchy and the price of stability, the reader is referred to [14].

1.1 Related work.

This above dynamics has been also modeled in the literature with *coloring games*. A coloring game is played on the conflict graph. Players must choose a color in order to construct a proper coloring of the graph, and the individual goal of each agent is to maximize the number of agents with the same color as she has. On a more theoretical side, coloring games have been introduced in [18] as a game-theoretic setting for studying the chromatic number in graphs. Specifically, the authors in [18] have shown that for every coloring game, there exists a Nash equilibrium where the number of colors is exactly the chromatic number of the graph. Since then, these games have been used many times, attracting attention in the study of information sharing and propagation in graphs [4, 7, 14]. Coloring games are an important subclass of the more general Hedonic games, of which several variations have been studied in the literature in order to model coalition formation under selfish preferences of the agents [10, 12, 15, 5, 8, 16]. We stress that while every coloring game has a Nash equilibrium that can be computed in polynomial-time [18], deciding whether a given Hedonic game admits a Nash equilibrium is NP-complete [1].

If the set of edges of the conflict graph is empty (edgeless conflict graph), there exists a unique k-stable partition namely that consists of the group of all the users. In [14], it is proved that there always exists a k-stable partition for any conflict graph, but that it is NP-hard to compute one (this result was also proved independently in [7]). Indeed, if k is equal to the number of users, a largest group in such a partition must be a maximum independent set of the conflict graph. In contrast, it can be computed a k-stable partition in polynomial time for every fixed $k \leq 3$, by using simple *better-response dynamics* [18, 7, 14]. In such an algorithm one does a k-deviation until there does not exist any one. That corresponds to the dynamics of formation of groups that we study in this work for larger values of k.

1.2 Additional related work and our results.

In this paper we are interested in analyzing in this simple model the convergence of the dynamics with k-deviations, in particular in the worst case. It has been proved implicitly in [14] that the dynamics always converges within at most $\mathcal{O}(2^n)$ steps. Let $L(k, G^-)$ be the size of a longest sequence of k-deviations on a conflict graph G^-. We first observe that the maximum value, denoted $L(k,n)$, of $L(k, G^-)$ over all the graphs with n vertices is attained on the edgeless conflict graph $G^{\emptyset}$ of order n. Prior to this work, no lower bound on $L(k,n)$ was known, and the analysis was limited to potential function that only applies when $k \leq 3$ [7, 14] giving upper bounds of $O(n^2)$ in the case $k = 1, 2$ and $O(n^3)$ in the case $k = 3$. In order to go further in our analysis, the key observation is that when the conflict graph is edgeless, the dynamics depends only of the size of the groups of the partitions generated. Following [3], let an integer partition of $n \geq 1$, be a non-increasing sequence of integers $Q = (q_1, q_2, \ldots, q_n)$ such that $q_1 \geq q_2 \geq \ldots \geq q_n \geq 0$ and $\sum_{i=1}^{n} q_i = n$. If we rank the groups by non increasing order of their size, there is a natural relation between partition in groups and integer partitions (the size q_i of the group X_i corresponding to the integers q_i of the partition of n). Using this relation, we prove in Section 3 that the better response dynamics algorithm reaches a stable partition in p_n steps, where $p_n = \Theta((e^{\pi\sqrt{2n/3}})/n)$ denotes the number of integer partitions. This is already far less than 2^n, which was shown to be the best upper bound that one can obtain for $k \geq 4$ when using an additive potential function [14].

Table 1 summarizes our contributions described below.

- For $k = 1, 2$, we refine the relation between partitions into groups and integer partitions as follows.
  - In the case $k = 1$ (Section 4.1), we prove that there is a one to one mapping between sequences of 1-deviations in the edgeless conflict graph and chains in the dominance lattice of integer partitions. Then, we use the value of the longest chain in this dominance lattice obtained in [9] to determine exactly $L(1,n)$. More precisely, if $n = \frac{m(m+1)}{2} + r$, with $0 \leq r \leq m$, $L(1,n) = 2\binom{m+1}{3} + mr$. The latter implies in particular $L(1,n)$ is of order $\mathcal{O}(n^{\frac{3}{2}})$, thereby improving the previous bound $\mathcal{O}(n^2)$.

k	Prior to our work	Our results	
1	$\mathcal{O}(n^2)$ [14]	exact analysis, which implies $L(1,n) \sim \frac{(2n)^{3/2}}{3}$	Theorem 6
2	$\mathcal{O}(n^2)$ [14]	exact analysis, which implies $L(2,n) \sim \frac{(2n)^{3/2}}{3}$	Theorem 9
1-2	$\mathcal{O}(n\alpha(G^-))$ [18]	$L(k,G^-) = \Omega(n\alpha(G^-))$ for some G^- and $\alpha(G^-) = \mathcal{O}(\sqrt{n})$	Theorem 12
3	$\mathcal{O}(n^3)$ [7, 14]	$L(3,n) = \Omega(n^2)$	Theorem 13
≥ 4	$\mathcal{O}(2^n)$ [14]	$L(k,n) = \Omega(n^{\Theta(\ln(n))})$, $L(k,n) = \mathcal{O}(\exp(\pi\sqrt{2n/3})/n)$	Theorem 14

Table 1 Previous bounds and results we obtained on $L(k,n)$ and $L(k,G^-)$.

- In Section 4.2, we prove that any 2-deviation can be "replaced" (in some precise way) either by one or two 1-deviations, and so, $L(2,n) = L(1,n)$.
- For $k = 1, 2$ and a general conflict graph G^-, the value of $L(k,G^-)$ depends on the independence number $\alpha(G^-)$ (cardinality of a largest independent set) of the conflict graph. In [18] it was proved that the convergence of the dynamics is in $\mathcal{O}(n\alpha(G^-))$. In the case of edgeless conflict graph, we have seen that $L(1,n) = \Omega(n^{3/2})$ and so the preceding upper-bound was not tight. So we inferred that the convergence of the dynamics was in $\mathcal{O}(n\sqrt{\alpha(G^-)})$. Yet in fact we prove in Section 4.3 that, for any $\alpha(G^-) = \mathcal{O}(\sqrt{n})$, there exists a conflict graph G^- with n vertices and independence number $\alpha(G^-)$ for which we need a sequence of at least $\Omega(n\alpha(G^-))$ 1-deviations to reach a stable partition. *For the wedding's example of the foreword, $\alpha(G^-) = 4$ and so the sequence is linearly bounded.*

- Finally, our main contribution is obtained for $k \geq 3$. Prior to our work, it was known that $L(3,n) = \mathcal{O}(n^3)$, which follows from another application of the potential function method [14]. But nothing proved that $L(3,n) > L(2,n)$, and in fact it was conjectured in [7] that both values are equal. In Section 5.2, we prove (Theorem 13) that $L(3,n) = \Omega(n^2)$ and thus we show for the first time that deviations can delay convergence and that the gap between $k = 2$ and $k = 3$ obtained from potential function is indeed justified. It was also conjectured in [14] that $L(k,n)$ was polynomial in n for k fixed. In Section 5.1 we disprove this conjecture and prove in Theorem 14 that $L(4,n) = \Omega(n^{\Theta(\ln(n))})$. This shows that 4-deviations are responsible for a sudden complexity increase, as no polynomial bounds exist for $L(4,n)$. *This explains why in the foreword it takes an exponential time for the organizers of FUN to schedule the groups.*

2 Notations

Conflict graph. We refer to [2] for standard graph terminology. For the remaining of the paper, we suppose that we are given a *conflict graph* $G^- = (V,E)$ where V is the set of vertices (called users or players in the introduction) and edges represent the incompatibility relation (*i.e.*, an edge means that the two users are enemies). The number of vertices is denoted by $n = |V|$. The independence number of G^-, denoted $\alpha(G^-)$, is the maximum cardinality of an independent set in G^-. In particular, if $\alpha(G^-) = n$ then the conflict graph is edgeless and we denote it by $G^\emptyset = (V, E = \emptyset)$ and call it the empty graph.

Partitions and utilities. We consider any partition $P = X_1, \ldots, X_i, \ldots, X_n$ of the vertices into n independent sets X_i called groups (colors in coloring games), with some of them being possibly empty. In particular, two enemies are not in the same group. We rank the groups by non increasing size, that is $|X_i| \geq |X_{i+1}|$. For any $1 \leq i \leq n$ and for any

$v \in X_i$, the *utility* of vertex v is the number of other vertices in the same group as it, that is $|X_i| - 1$.

We use in our proofs two alternative representations of the partition P. The *partition vector* associated to P is defined as $\overrightarrow{\Lambda}(P) = (\lambda_n(P), \ldots, \lambda_1(P))$, where $\lambda_i(P)$ is the number of groups of size i. The *integer partition* associated to P is defined as $Q = (q_1, q_2, \ldots, q_n)$ such that $q_1 \geq q_2 \geq \ldots \geq q_n \geq 0$ and $\sum_{i=1}^{n} q_i = n$, where $q_i = |Xi|$.

In the example of Figure 1(a) we have a partition P of the 12 vertices into 4 groups each of size 3 and so $\lambda_3(P) = 4$ and $\lambda_i(P) = 0$ for $i \neq 3$; in other words $\overrightarrow{\Lambda}(P) = (0,0,0,0,0,0,0,0,0,4,0,0)$. The integer partition $Q(P) = (3,3,3,3,0,0,0,0,0,0,0,0)$. In the example of Figure 1(b) we have a partition P' of the 12 vertices into one group of size 4 and 4 groups each of size 2 and so $\lambda_4(P') = 1$, $\lambda_2(P') = 4$ and $\lambda_i(P') = 0$ for $i \neq 2, 4$; in other words $\overrightarrow{\Lambda}(P) = (0,0,0,0,0,0,0,0,1,0,4,0)$. The integer partition $Q(P') = (4,2,2,2,2,0,0,0,0,0,0,0)$.

k-deviations and k-stability. We can think of a k-deviation as a move of at most k vertices which leave the groups to which they belong in P, to join another group (or create a new group) with the necessary condition that each vertex strictly increases its utility, thereby leading to a new partition P'. A *k-stable partition* is simply a partition for which there exists no k-deviation. We write $L(k, G^-)$, resp. $L(k, n)$, for the length of a longest sequence of k-deviations to reach a stable partition in G^-, resp. in any conflict graph with n vertices. Recall that we start with the partition consisting of n groups of size 1, that is, $\overrightarrow{\Lambda}(P) = (\ldots, 0, 0, 0, n)$.

We next define a natural vector representation for k-deviations. The *difference vector* $\overrightarrow{\varphi}$ associated to a k-deviation φ from P to P' is equal to $\overrightarrow{\varphi} = \overrightarrow{\Lambda}(P') - \overrightarrow{\Lambda}(P)$. In concluding this section, we define the difference vectors for some of the k-deviations used in our proofs:

- $\overrightarrow{\alpha}[p,q]$, the 1-deviation where a vertex leaves a group of size $q+1$ for a group of size $p-1$ (valid when $p \geq q+2$). In that case $\alpha_p = 1, \alpha_{p-1} = -1, \alpha_{q+1} = -1, \alpha_q = -1$, and $\alpha_i = 0$ for any $i \notin \{q, q+1, p-1, p\}$ (we omit for ease of reading the brackets $[p,q]$).
- $\overrightarrow{\gamma}[p]$, the 3-deviation where one vertex in each of 3 groups of size $p-1$ moves to a group of size $p-3$ to form a new group of size p (valid if there are at least 3 groups of size $p-1$ and one of size $p-3$). In that case $\gamma_p = 1, \gamma_{p-1} = -3, \gamma_{p-2} = 3, \gamma_{p-3} = -1$, and $\gamma_i = 0$ for any $i \notin \{p-3, p-2, p-1, p\}$.
- $\overrightarrow{\delta}[p]$, the 4-deviation where one vertex in each of 4 groups of size $p-1$ moves to a group of size $p-4$ to form a new group of size p (valid if there are at least 4 groups of size $p-1$ and one of size $p-4$). In that case $\delta_p = 1, \delta_{p-1} = -4, \delta_{p-2} = 4, \delta_{p-4} = -1$, and $\delta_i = 0$ for any $i \notin \{p-4, p-2, p-1, p\}$. As an example, the move from the partition of Figure 1(a) to the partition of Figure 1(b), is a 4-deviation with difference vector $\overrightarrow{\delta[4]}$.

3 Preliminary results

In [14], the authors prove that there always exists a k-stable partition, but that it is NP-hard to compute one (this result was also proved independently in [7]). In contrast, it can be computed a k-stable partition in polynomial time for every fixed $k \leq 3$, by using simple *better-response dynamics* [18, 7, 14]. The latter results question the role of the value of k in the complexity of computing stable partitions.

Formally, a better-response dynamics proceeds as follows. We start from the trivial partition P_1 consisting of n groups with one vertex in each of them. In particular, the partition vector $\overrightarrow{\Lambda}(P_1)$ is such that $\lambda_1(P_1) = n$ and, for all other $j \neq 1$, $\lambda_j(P_1) = 0$.

Provided there exists a k-deviation with respect to the current partition P_i, we pick any one of these k-deviations φ and in so doing we obtain a new partition P_{i+1}. If there is no k-deviation, the partition P_i is k-stable. An algorithmic presentation is given in Algorithm 1.

Dynamics of the system (Algorithm 1)

Input: a positive integer $k \geq 1$, and a conflict graph G^-.
Output: a k-stable partition for G^-.

1: Let P_1 be the partition composed of n singletons groups.
2: Set $i = 1$.
3: **while** there exists a k-deviation for P_i **do**
4: Set $i = i + 1$.
5: Choose one k-deviation and compute the partition P_i after this k-deviation.
6: Return the partition P_i.

We now prove in Proposition 1 that better-response dynamics can be used for computing a k-stable partition *for every fixed* $k \geq 1$ (but not necessarily in polynomial time). It shows that for every fixed $k \geq 1$, the problem of computing a k-stable partition is in the complexity class PLS (Polynomial Local Search), that is conjectured to lie strictly between P and NP [13]. Recall that the problem becomes NP-hard when k is part of the input.

▶ Proposition 1. For any $k \geq 1$, for any conflict graph G^-, Algorithm 1 converges to a k-stable partition.

Proof. Let P_i, P_{i+1} be two partitions for G^- such that P_{i+1} is obtained from P_i after some k-deviation φ. Let S be the set of vertices which move ($|S| \leq k$) and let j be the size of the group they join ($j = 0$ if they create a new group). Then, the new group obtained has size $p = j + |S|$. Note that all the vertices of S have increased their utilities and so, they belonged in P_i to groups of size $< p$. Therefore, the coordinates of the difference vector $\overrightarrow{\varphi}$ satisfy $\varphi_p = 1$ and $\varphi_j = 0$ for $j > p$, and so $\overrightarrow{\Lambda}(P_i) <_L \overrightarrow{\Lambda}(P_{i+1})$ where $<_L$ is the lexicographical ordering. Finally, as the number of possible partition vectors is finite, we obtain the convergence of Algorithm 1. ◀

An instrumental observation for our next proofs is the following:

▶ Observation 1. $L(k, n)$ is always attained on the empty conflict graph $G^{\emptyset}$ of order n.

Indeed, any sequence of k-deviations on a conflict graph G^- is also a sequence in the empty conflict graph with the same vertices. Note that the converse is not true as it can happen that some moves allowed in the empty conflict graph are not allowed in G^- as they bring two enemies in the same group.

Recall that we can associate to any partition $P = X_1, \dots, X_i, \dots, X_n$ of the vertices the integer partition $Q = (q_1, q_2, \dots, q_n)$ such that $q_1 \geq q_2 \geq \dots \geq q_n \geq 0$ and $\sum_{i=1}^{n} q_i = n$ by letting $q_i = |X_i|$. The converse is not true in general; as example it suffices to consider a partition with $q_1 > \alpha(G^-)$. However the converse is true when the conflict graph is empty; indeed it suffices to associate to an integer partition any partition of the vertices obtained by putting in the group X_i a set of q_i vertices .

We can now use the value of the number p_n of the number of integer partitions (see [11]) to obtain the following proposition which follows from Proposition 1.

▶ Proposition 2. Algorithm 1 reaches a stable partition in at most $p_n = \Theta((e^{\pi\sqrt{\frac{2n}{3}}})/n)$ steps.

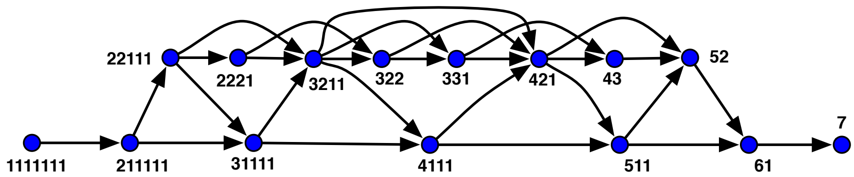

Figure 2 The lattice of integer partitions for $n = 7$.

Note that this is already far less than 2^n, which was shown to be the best upper bound that one can obtain for $k \geq 4$ when using an additive potential function [14].

4 Analysis for k $\leq$ 2

In [14], the authors proved that for $k \leq 2$, Algorithm 1 converges to a stable partition in at most a quadratic time (by using a potential function). Indeed when performing a 1-deviation $\overrightarrow{\alpha}[p, q]$, a vertex moves from a group of size $q+1$ to a group of size $p-1$ (with $p \geq q+2$); the utility of this vertex increases by $p-q-1$, the utility of the q other vertices of the group of size $q+1$ decreases by 1, while the utility of the vertices of the group of size $p-1$ increases by 1 and so the global utility increases by $2p-2q-2 \geq 2$ as $p \geq q+2$. Furthermore, in a k-stable partition, the utility of a vertex is at most $n-1$ and the global utility is at most $n(n-1)/2$ and so $L(k, n) = O(n^2)$.

In the next subsections we improve this result as we completely solve this case and give the exact (non-asymptotic) value of $L(k, n)$ when $k \leq 2$. The gist of the proof is to use a partial ordering that was introduced in [3], and is sometimes called the dominance ordering.

4.1 Exact analysis for k = 1 and empty conflict graph

In [3] the author has defined an ordering over the integer partitions, sometimes called the dominance ordering which creates a lattice of integer partitions. This ordering is a direct application of the theory of majorization to integer partitions [17].

▶ **Definition 3.** (dominance ordering) Given two integer partitions of $n \geq 1$, $Q = (q_1, q_2, \ldots, q_n)$ and $Q' = (q'_1, q'_2, \ldots, q'_n)$, we say that Q' dominates Q if $\sum_{j=1}^{i} q'_j \geq \sum_{j=1}^{i} q_j$, for all $1 \leq i \leq n$.

The example of Figure 2 shows the dominance lattice for $n = 7$. We did not write in the figure the integers equal to 0. Now we will show that there is a one to one mapping between chains in the dominance lattice and sequences of 1-deviations in the empty conflict graph.

The two next lemmas show that there is a one to one mapping between chains in the dominance lattice and sequences of 1-deviations in the empty conflict graph.

▶ **Lemma 4.** *Let P be a partition of the vertices and P' be the partition obtained after a 1-deviation φ. Then, the integer partition $Q' = Q(P')$ dominates $Q = Q(P)$.*

Proof. In the 1-deviation φ a vertex v moves from a group X_k to a group X_j with $q_j = |X_j| \geq q_k = |X_k|$. W.l.o.g. we can suppose that the groups (ranked in non increasing order of size) are ranked in a such a way that X_j is the first group with size $|X_j|$ and X_k the last group with size $|X_k|$. Thus, the integer partition $Q(P)$ associated to P satisfies $q_1 \geq q_2 \ldots \geq q_{j-1} > q_j \geq q_{j+1} \ldots \geq q_k > q_{k+1} \ldots \geq q_n$. After the move the groups of P' are the same as those of P except we have replaced X_j with the group $X_j \cup v$ and X_k with $X_k - v$. Therefore the integer partition Q' associated to P' has the same elements as Q

except $q'_j = q_j + 1$ and $q'_k = q_k - 1$ and so Q' dominates Q. Note that this lemma holds for any conflict graph. ◀

In the case $n = 7$, consider the partition P with one group of size 3, one of size 2 and two of size 1. The integer partition associated to P is $Q = (3, 2, 1, 1, 0, 0, 0)$. Let φ be the 1-deviation where a vertex in the group of size 1 moves to the group of size 3. We obtain the partition P' with one group of size 4, one of size 2 and one of size 1. The integer partition associated to P' is $Q' = (4, 2, 1, 0, 0, 0, 0)$ which dominates Q.

▶ **Lemma 5.** *Let $G^{\emptyset}$ be the empty conflict graph and let Q, Q' be two integer partitions of $n = |V|$ such that Q' dominates Q. For any partition P associated to Q, there exists another partition P' associated to Q' such that P' is obtained from P by doing a sequence of 1-deviations.*

Proof. As proved in [3], we have that if Q' dominates Q then there is a finite sequence of integer partitions $Q^0, \ldots, Q^r, \ldots, Q^s$,with $Q = Q^0$ and $Q' = Q^s$ such that for each $0 \leq r < s$, Q^{r+1} dominates Q^r and differs from it only in two elements j_r and k_r with $q^{r+1}_{j_r} = q^r_{j_r} + 1$ and $q^{r+1}_{k_r} = q^r_{k_r} - 1$.

The proof is now by induction on r, starting from any partition $P^0 = P$ associated to Q. For $r > 0$, we consider the partition P^r associated to Q^r. Recall that Q^r and Q^{r+1} differ only in the two groups X_{j_r} and X_{k_r}. As $q^{r+1}_{j_r} = q^r_{j_r} + 1$ and $q^{r+1}_{k_r} = q^r_{k_r} - 1$, P^{r+1} can be obtained from P^r by moving a vertex from X_{k_r} to X_{j_r}. This move is valid as the conflict graph is empty. (Note that the lemma is not valid for a general conflict graph.) ◀

As an example, consider the two integer partitions $Q = (2, 2, 2, 1, 0, 0, 0)$ and $Q' = (5, 1, 1, 0, 0, 0, 0)$ where Q' dominates Q. The sequence of integer partitions is $Q_0 = Q$, $Q_1 = (3, 2, 1, 1, 0, 0, 0)$, $Q_2 = (4, 1, 1, 1, 0, 0, 0)$, $Q_3 = (5, 1, 1, 0, 0, 0, 0)$. Partition P^1 is obtained from P^0 by moving a vertex of a group of size 2 to another group of size 2. Then, P^2 is obtained by moving a vertex of the group of size 2 to the group of size 3 and P' is obtained from P^2 by moving a vertex of one group of size 1 to that of size 4.

In summary we conclude that a sequence of 1-deviations with an empty conflict graph corresponds to a chain of integer partitions, and vice versa. Therefore, by Observation 1, the length of the longest sequence of 1-deviations with an empty conflict graph is the same as the length of the longest chain in the dominance lattice of integer partitions. Since it has been proven in [9] that for $n = \frac{m(m+1)}{2} + r$, the longest chain in the Dominance Lattice has length $2\binom{m+1}{3} + mr$, we obtain the *exact* value for $L(1, n)$.

▶ **Theorem 6.** *Let m and r be the unique non negative integers such that $n = \frac{m(m+1)}{2} + r$, and $0 \leq r \leq m$. Then, $L(1, n) = 2\binom{m+1}{3} + mr$.*

We note that the proof in [9] is not straightforward. One can think that the longest chain is obtained by taking among the possible 1-deviations the one which leads to the smallest partition in the lexicographic order. Unfortunately this is not true. Indeed let $n = 9$. After 6 steps we get the integer partition $(3, 3, 2, 1, 0, 0, 0, 0, 0)$. Then, by choosing the 1-deviation that gives the smallest partition (in the lexicographic order), we get the partition $(3, 3, 3, 0, 0, 0, 0, 0, 0)$ and then $(4, 3, 2, 0, 0, 0, 0, 0, 0)$. But there is a longer chain of length 3 from $(3, 3, 2, 1, 0, 0, 0, 0, 0)$ to $(4, 3, 2, 0, 0, 0, 0, 0, 0)$ namely $(4, 2, 2, 1, 0, 0, 0, 0, 0)$, $(4, 3, 1, 1, 0, 0, 0, 0, 0)$, $(4, 3, 2, 0, 0, 0, 0, 0, 0)$. However the proof in [9] implies that the following simple construction works for any n.

▶ Proposition 7. A longest sequence of 1-deviations in the empty conflict graph is obtained by choosing, at a given step, among all the possible 1-deviations, any one of which leads to the smallest increase of the global utility.

Proof. We need to introduce the terminology of [9] (this will not be used elsewhere in the paper). Note that they consider their chains starting from the end, and so, we need to reverse the steps in their construction in order to make them correspond with 1-deviations.

- A V-step is corresponding to a user leaving her group of size p for another group of size p, thereby increasing her utility from $p-1$ to p; in other words, the deviation vector of such 1-deviation is $\overrightarrow{\alpha}[p+1, p-1]$ for some p.
- An H-step is corresponding to a user leaving her group X_i of size p for another group X_j of size $p \leq |X_j| \leq p+1$, but *only* if there is no other group of size p than X_i and (possibly) X_j; in particular, if groups are ordered by decreasing size, this means that $j = i-1$. The deviation vector of an H-step is either $\overrightarrow{\alpha}[p+1, p-1]$ (and then, X_i, X_j are the only two groups of size p) or $\overrightarrow{\alpha}[p+2, p-1]$ (and then X_i is the unique group of size p). Furthermore, note that an H-step can also be a V-step.

The relationship between V-steps, H-steps and our construction is as follows. At every 1-deviation, the global utility has to increase by at least two, and this is attained if and only if the deviation vector is $\overrightarrow{\alpha}[p+1, p-1]$ for some p; equivalently, this move is corresponding to a V-step. Furthermore, if no such a move is possible, then the global utility has to increase by at least four, and this is attained if and only if the deviation vector is $\overrightarrow{\alpha}[p+2, p-1]$ for some p; in such a case, there cannot exist any other group of size p than the one involved in the 1-deviation (otherwise, a V-step would have been possible), hence this move is corresponding to a H-step. As proved, *e.g.*, by Brylawski [3], starting from any integer partition with at least two summands (*i.e.*, $q_n \neq 1$), it is always possible to perform one of the two types of move defined above (these two types of move actually correspond to the two cases when an integer partition can cover another one). Therefore, our strategy leads to a sequence of 1-deviations where all the moves correspond to either a V-step or (only if the first type of move is not possible) to an H-step. By a commutativity argument (Lemma 3 in [9]) it can be proved that as soon as no move $\overrightarrow{\alpha}[p+1, p-1]$ (corresponding to V-steps) is possible for any p, every ulterior move of this type will correspond to both a V-step and an H-step simultaneously (*i.e.*, the deviation vector will be $\overrightarrow{\alpha}[p+1, p-1]$ for some p, and there will be no other groups of size p than the two groups involved in the 1-deviation). Therefore, the sequences we obtain are corresponding to a particular case of the so-called HV-chains in [9]. Finally, the main result in [9] is that every HV-chain is of maximum length. ◀

In what follows, we will reuse part of the construction in [9] for proving Theorem 10.

4.2 Analysis for k = 2

Interestingly we will prove that any 2-deviation can be replaced either by one or two 1-deviations and so we will prove in Theorem 9 that $L(2, n) = L(1, n)$.

▶ Claim 8. If the conflict graph G^- is empty, then any 2-deviation can be replaced either by one or two 1-deviations

Proof. Consider a 2-deviation which is not a 1-deviation. In that case case two vertices u_i and u_j leave their respective group X_i and X_j (which can be the same) to join a group X_k. Let $|X_i| \geq |X_j|$; in order for the utility of the vertices to increase, we should have $|X_k| \geq |X_i| - 1$ ($\geq |X_j| - 1$).

- Case 1: $|X_k| \geq |X_j|$. In that case the 2-deviation can be replaced by a sequence of two 1-deviations where firstly a vertex u_j leaves X_j to join X_k and then a vertex u_i leaves X_i to join the group $X_k \cup u_j$ whose size is now at least that of X_i.

- Case 2: $|X_k| = |X_i| - 1 = |X_j| - 1 = p - 2$ and $X_i = X_j$. In that case the effect of the 2-deviation is to replace the group X_i of size $p-1$ with a group of size $p-3$ and to replace the group X_k of size $p-2$ with a group of size p. Said otherwise, the difference vector $\overrightarrow{\varphi}$ associated to the 2-deviation has as non null coordinates $\varphi_p = 1, \varphi_{p-1} = -1, \varphi_{p-2} = -1, \varphi_{p-3} = 1$. We obtain the same effect by doing the 1-deviation $\overrightarrow{\alpha}[p-1, p-2]$ where a vertex leaves X_k to join X_i.
- Case 3: $|X_k| = |X_i| - 1 = |X_j| - 1 = p - 2$ and $X_i \neq X_j$. In that case the effect of the 2-deviation is to replace the 2 groups X_i and X_j of size $p-1$ with two groups of size $p-2$ and to replace the group X_k of size $p-2$ with a group of size p. Said otherwise, the difference vector $\overrightarrow{\varphi}$ associated to the 2-deviation has as non null coordinates $\varphi_p = 1, \varphi_{p-1} = -2, \varphi_{p-2} = 1$. We obtain the same effect by doing the 1-deviation $\overrightarrow{\alpha}[p-1, p-1]$ where a vertex leaves X_j to join X_i.

Note that the fact G^- is empty is needed for the proof. Indeed, in the case 2, it might happen that all the vertices of X_k have some enemy in X_i and so the 1-deviation we describe is not valid. Similarly, in case 3, it might happen that all the vertices of X_i have some enemy in X_j and so the 1-deviation we describe is not valid. ◇

▶ **Theorem 9.** $L(2, n) = L(1, n)$.

Proof. Clearly, $L(2, n) \geq L(1, n)$ as any 1-deviation is also a 2-deviation. By Observation 1, the value of $L(2, n)$ is obtained when the conflict graph G^- is empty. In that case, Claim 8 implies that $L(2, n) \leq L(1, n)$. ◀

4.3 Analysis for k ≤ 2 and a general conflict graph

Using the potential function introduced at the beginning of this section, Panagopoulou and Spirakis ([18]) proved that for every conflict graph G^- with independence number $\alpha(G^-)$, the convergence of the dynamics is in $\mathcal{O}(n\alpha(G^-))$. Indeed as we have seen each 1-deviation increases the global utility by at least 2. But the global utility of a stable partition is at most $n(\alpha(G^-) - 1)$ as the groups have maximum size $\alpha(G^-)$. If the conflict graph is empty we have seen that $L(1, n) = \Theta(n^{3/2})$ that is in that case $\mathcal{O}(n\sqrt{\alpha(G^-)})$. This leads one of us ([6], page 131) to conjecture that in the case of 1-deviations the worst time of convergence of the dynamics is $\mathcal{O}(n\sqrt{\alpha(G^-)})$. We disprove the conjecture by proving the following theorem:

▶ **Theorem 10.** *For $n = \binom{m+1}{2}$, there exists a conflict graph G^- with $\alpha(G^-) = m = \Theta(\sqrt{n})$ and a sequence of $\binom{m+1}{3}$ valid 1-deviations, that is a sequence of $\Omega(n^{\frac{3}{2}}) = \Omega(n\alpha(G^-))$ 1-deviations.*

Proof. We will use part of the construction of Greene and Kleitman ([9]). Namely they prove that, if $n = \binom{m+1}{2}$, there is a sequence of $\binom{m+1}{3}$ 1-deviations transforming the partition P_1 consisting of n groups each of size 1 (the coordinates of $\overrightarrow{\Lambda}(P_1)$ satisfy $\lambda_1 = n$) into the partition P_m consisting of m groups, one of each possible size i for $1 \leq i \leq m$ (the coordinates of $\overrightarrow{\Lambda}(P_m)$ satisfy $\lambda_i = 1$ for $1 \leq i \leq m$). Furthermore they prove that the moves used are V-steps (see the proof of proposition 7 that is $\overrightarrow{\alpha}[p+1, p-1]$ for some p (one vertex leaves a group of size p to join a group of the same size p). One can note that in such a move the utility increases only by 2 and as the total utility of P_m is $\sum_{i=1}^{m} i(i-1) = (m+1)m(m-1)/3$ the number of moves is $(m+1)m(m-1)/6$.

The conflict graph of the counterexample will consist of m complete graphs $K^j, 1 \leq j \leq m$ where K^j has exactly j vertices. An independent set is therefore formed by taking at most one vertex in each K^j and $\alpha(G^-) = m$. We will denote the elements of K^j by $\{x_i^j\}$

with $1 \leq i \leq j \leq m$. The group of P_m of size i will be $X_i = \bigcup x_i^j$ with $m+1-i \leq j \leq m$. So these groups are independent sets.

Recall that $n = m(m+1)/2$. For each $p, 1 \leq p \leq m$ let us denote by P_p the partition consisting of 1 group of each size i for $1 \leq i \leq p$ and $n - p(p+1)/2$ groups of size 1 (said otherwise the coordinates of $\overrightarrow{\Lambda}(P_p)$ satisfy $\lambda_i = 1$ for $2 \leq i \leq p$ and $\lambda_1 = 1 + n - p(p+1)/2$). We will now describe the sequence $\overrightarrow{\sigma}[p-1]$ of $p(p-1)/2$ 1-deviations which transform the partition P_{p-1} into P_p. One way to do the Greene-Keitman sequence is obtained by doing successively the sequences $\sum_{p=2}^{m} \overrightarrow{\sigma}[p-1]$. More precisely we will prove by induction the following fact:

▶ Claim 11. There exists a sequence $\overrightarrow{\sigma}[p-1]$ of $p(p-1)/2$ valid 1-deviations which transform the partition P_{p-1} into P_p such that after this sequence the group $X_i[p]$ of size i, $1 \leq i \leq p$ contains exactly the vertices $X_i[p] = \bigcup x_{i+m-p}^j$ with $m+1-i \leq j \leq m$.

Proof. (the reader can follow the construction on the example after)

We suppose we have built the sequence till $p-1$ and that, for $1 \leq i \leq p-1$, $X_i[p-1] = \bigcup x_{i+m-p+1}^j$ with $m+1-i \leq j \leq m$. In a first phase we consider the subpartition of $n-p+1$ elements obtained by removing the group $X_{p-1}[p-1]$. Namely, this above subpartition consists of the groups $X_i[p-1]$ for $1 \leq i \leq p-2$ and groups of size 1. In particular, the subpartition is isomorphic to P_{p-2} with $p-1$ singleton groups removed. Our construction ensure that these $p-1$ singleton groups that are missing are not used for $\overrightarrow{\sigma}[p-2]$. So, we can do the transformation $\overrightarrow{\sigma}[p-2]$ consisting of $(p-1)(p-2)/2$ valid moves on the partition of $n-p+1$ elements not contained in $X_{p-1}[p-1]$. It gives rise to the groups $X_i[p] = X_{i-1}[p-1] + x_{i+m-p}^{m+1-i}$. Note that at this stage we have two groups of size $p-1$, namely the original one $X_{p-1}[p-1]$ and the new one constructed $X_{p-1}[p]$. The second phase consists in doing $p-1$ successive 1-deviations with the vertex x_m^{m+1-p}. More precisely we move this vertex to the group $X_1[p]$ created in the first phase, then from this group to $X_2[p]$ and so on till $X_{p-2}[p]$ and finally from $X_{p-2}[p]$ to the original $X_{p-1}[p-1]$. The moves are valid as we move a vertex from K^{m+1-p} and the groups did not contain any vertex of this complete graph. Groups created in the first phase are eventually left unchanged as x_m^{m+1-p} joins this group and then leaves it. Finally we have constructed a new group $X_p[p] = X_{p-1}[p-1] \cup x_m^{m+1-p}$. The groups are exactly those described in the claim. ◇

To end the proof of Theorem 10, it suffices to note that the groups X_i form an independent set and that after $\sum_{p=2}^{m} \overrightarrow{\sigma}[p-1]$ we have obtained the desired groups of P_m which gives the counterexample. ◀

Example for $\mathbf{m = 4}$. (See Figure 3.)

- After $\overrightarrow{\sigma}[1]$, we have the 2 groups $X_2[2] = x_4^4 \cup x_4^3$ and $X_1[2] = x_3^4$.
- First phase of $\overrightarrow{\sigma}[2]$: we do the move of $\overrightarrow{\sigma}[1]$ on the vertices not in $X_2[2]$ and create the groups $X_2[3] = x_3^4 \cup x_3^3$ and $X_1[3] = x_2^4$.
- Second phase of $\overrightarrow{\sigma}[2]$: now we move x_4^2 to $X_1[3]$ and then from $X_1[3]$ to the original $X_2[2] = x_4^4 \cup x_4^3$, thereby creating the group $X_3[3] = x_4^4 \cup x_4^3 \cup x_4^2$.
- First phase of $\overrightarrow{\sigma}[3]$: we do the 3 moves of $\overrightarrow{\sigma}[2]$ on the vertices not in $X_3[3]$ and create the groups $X_3[4] = x_3^4 \cup x_3^3 \cup x_3^2$, $X_2[4] = x_2^4 \cup x_2^3$, $X_1[4] = x_1^4$.
- Second phase of $\overrightarrow{\sigma}[3]$: now we move x_4^1 to $X_1[4]$, then from $X_1[4]$ to $X_2[4]$ and finally from $X_2[4]$ to the original $X_3[3] = x_4^4 \cup x_4^3 \cup x_4^2$, thereby creating the group $X_4[4] = x_4^4 \cup x_4^3 \cup x_4^2 \cup x_4^1$.

We can prove a theorem analogous to Theorem 10 for any independence number $\alpha(G^-)$.

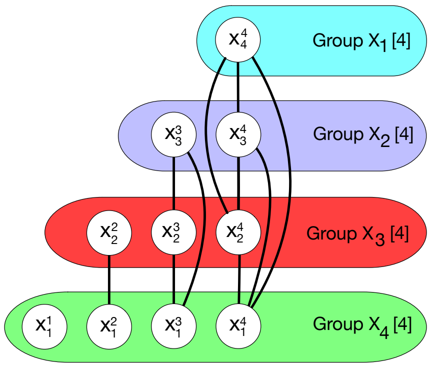

Figure 3 Illustration for Example 4.

▶ **Theorem 12.** *For any $\alpha = \mathcal{O}(\sqrt{n})$, there exists a conflict graph G^- with n vertices and independence number $\alpha(G^-) = \alpha$, and a sequence of at least $\Omega(n\alpha)$ 1-deviations to reach a stable partition.*

Proof. Let G_0^- be the graph of Theorem 10 for $m = \alpha$. G_0^- has $n_0 = \mathcal{O}(\alpha^2)$ vertices, independence number α, and furthermore there exists a sequence of $\Theta(\alpha^3)$ valid 1-deviations for G_0^-. Let G^- be the graph obtained by taking the complete join of $k = n/n_0$ copies of G_0^- (*i.e.*, we add all possible edges between every two copies of G_0^-). By construction, G^- has order $n = kn_0 = \mathcal{O}(n\alpha^2)$ and the same independence number α as G_0^-. Furthermore, there exists a sequence of $k\Theta(\alpha^3) = \Omega(n\alpha)$ valid 1-deviations for G^-. ◀

Note that in any 2-deviation the global utility increases by at least 2 and so the number of 2 deviations when the conflict graph has independence number $\alpha(G^-)$ is also at most $\mathcal{O}(n\alpha(G^-))$. This bound is attained by using only 1-deviations as proved in Theorem 12, which is also valid for $k = 2$.

5 Lower bounds for k $>$ 2

The classical dominance ordering does not suffice to describe all k-deviations as soon as $k \geq 3$. As noted before, there is only one k-stable partition P_{max} in the empty conflict graph $G^{\emptyset}$, namely the one consisting of one group of size n, with integer partition $Q_{max} = (n, 0, \ldots, 0)$ and partition vector $(1, 0, \ldots, 0)$. Let $d(Q)$ be the length of a longest sequence in the dominance lattice from the integer partition Q to the integer partition Q_{max}. For $k = 4$ let P be the partition consisting of 4 groups of size 4 and one group of size 1 with integer partition $Q = (4, 4, 4, 4, 1)$. Apply the 4-deviation where one vertex of each group of size 4 joins the group of size 1; it leads to the partition P' with integer partition $Q' = (5, 3, 3, 3, 3)$. Q is covered in the dominance lattice by the integer partition $(5, 4, 4, 3, 1)$ while Q' is at distance 3 from it via $(5, 4, 3, 3, 2)$ and $(5, 4, 4, 2, 2)$ and so $d(Q') = d(Q) + 2$.

Prior to our work, it was known $L(3, n) = O(n^3)$ ([14]). But nothing proved that $L(3, n) > L(2, n)$, and in fact it was conjectured in [7] that both values are equal. Theorem 13 proves for the first time that deviations can delay convergence and that the gap between $k = 2$ and $k = 3$ obtained from potential function is indeed justified. It was also conjectured in [14] that $L(k, n)$ was polynomial in n for k fixed. We disprove this conjecture and prove in Theorem 14 a much more significant result: 4-deviations are responsible for a sudden complexity increase, as we prove that no polynomial bounds exist for $L(4, n)$.

▶ **Theorem 13.** $L(3, n) = \Omega(n^2)$.

▶ **Theorem 14.** $L(4, n) = \Omega(n^{\Theta(\ln(n))})$.

The main idea of the proofs consists in doing repeated shifted sequences (called cascades) of deviations similar to the ones given in the example above. The proof of Theorems 13 can be found in the appendix (Section 5.2). In the next section, we give the the proof of Theorem 14 for $k = 4$. We use sequences (cascades) of 4-deviations, called $\delta[p]$, and various additional tricks such that the repetition of the process by using cascades of cascades. Our motivation for using $\delta[p]$ as a basic building block for our construction is that it is the only type of 4-deviation which decreases the global utility.

5.1 Case k = 4. Proof of Theorem 14

Definition of $\delta[p]$: Consider a partition P containing at least 4 groups of size $p-1$ and 1 group of size $p-4$. In the 4-deviation $\delta[p]$ one vertex in each of the 4 groups of size $p-1$ moves to the group of size $p-4$ to form a new group of size p. The example given at the beginning of this section corresponds to the case $p = 5$. The coordinates of the associated difference vector (where we omit the bracket $[p]$ for ease of reading) are:

...	δ_p	δ_{p-1}	δ_{p-2}	δ_{p-3}	δ_{p-4}	...
...0	1	-4	4	0	-1	0...

Figure 4 gives a visual description of these cascades. Here we start with a sequence of t 4-deviations $\delta[p]$ represented by black rectangles ($t = 16$ in the figure). The cascade so obtained, called $\overrightarrow{\delta}^1[p, t]$, is represented in red. Then we do $(t-2)$ such cascades represented by red rectangles getting the cascade $\overrightarrow{\delta}^2[p, t-2]$ represented in yellow which contains $224 (= 16 \cdot 14)$ 4- deviations. We apply some 1-deviations to get a deviation called $\overrightarrow{\tau}^2[p]$ with the so-called Nice Property enabling us to do recursive constructions. We do a cascade of these $\overrightarrow{\tau}^2[p]$ (shifted by 2) represented by yellow rectangles getting the blue cascade called $\overrightarrow{\tau}^3[p]$. We do a cascade of these $\overrightarrow{\tau}^3[p]$ (shifted by 3) represented by blue rectangles getting the green cascade called $\overrightarrow{\tau}^4[p]$ and we finally do a cascade of these $\overrightarrow{\tau}^4[p]$ (shifted by 5) represented by green rectangles getting the grey cascade called $\overrightarrow{\tau}^5[p]$. The reader has to realize that, in this example, $\overrightarrow{\tau}^5[p]$ contains 3 cascades $\overrightarrow{\tau}^4[p]$ each containing 5 cascades $\overrightarrow{\tau}^3[p]$ each consisting of 7 cascades $\overrightarrow{\tau}^2[p]$. Altogether the cascade $\overrightarrow{\tau}^5[p]$ of this example contains 23520 4-deviations $\delta[p]$.

The cascade $\overrightarrow{\delta}^1[p, t]$: we first do a cascade consisting of a sequence of t shifted 4-deviations $\delta[p], \delta[p-1], \ldots, \delta[p-t+1]$, for some parameter t which will be chosen later to give the maximum number of 4-deviations.

The reader can follow the construction in Table 2 with $t = 7$. The coordinates $\overrightarrow{\delta}^1[p, t]$, are given in Claim 15 and Table 10. We note that there are lot of cancellations and only 8 non zero coordinates. Indeed consider the groups of size $p-i$ for $4 \leq i \leq t-1$; we have deleted such a group when doing the 4-deviation $\overrightarrow{\delta}[p+4-i]$, then created 4 such groups with $\overrightarrow{\delta}[p+2-i]$, then deleted 4 such groups with $\overrightarrow{\delta}[p+1-i]$, and finally created one with $\overrightarrow{\delta}[p-i]$. The reader can follow these cancellations in Table 2 for $i = 4, 5, 6$. The variation of the number of groups of a given size $p-i$ (which correspond to the coordinate δ^1_{p-i}) is obtained by summing the coefficients appearing in the corresponding column and so is 0 for $p-4, p-5, p-6$.

▶ Claim 15. For $3 \leq t \leq p-3$, the coordinates of the cascade $\overrightarrow{\delta}^1[p, t] = \sum_{i=0}^{t-1} \overrightarrow{\delta}[p-i]$ satisfy: $\delta^1_p = 1$, $\delta^1_{p-1} = -3$, $\delta^1_{p-2} = 1$, $\delta^1_{p-3} = 1$, $\delta^1_{p-t} = -1$, $\delta^1_{p-t-1} = 3$, $\delta^1_{p-t-2} = -1$, $\delta^1_{p-t-3} = -1$, and $\delta^1_j = 0$ for all the others j (see Table 10).

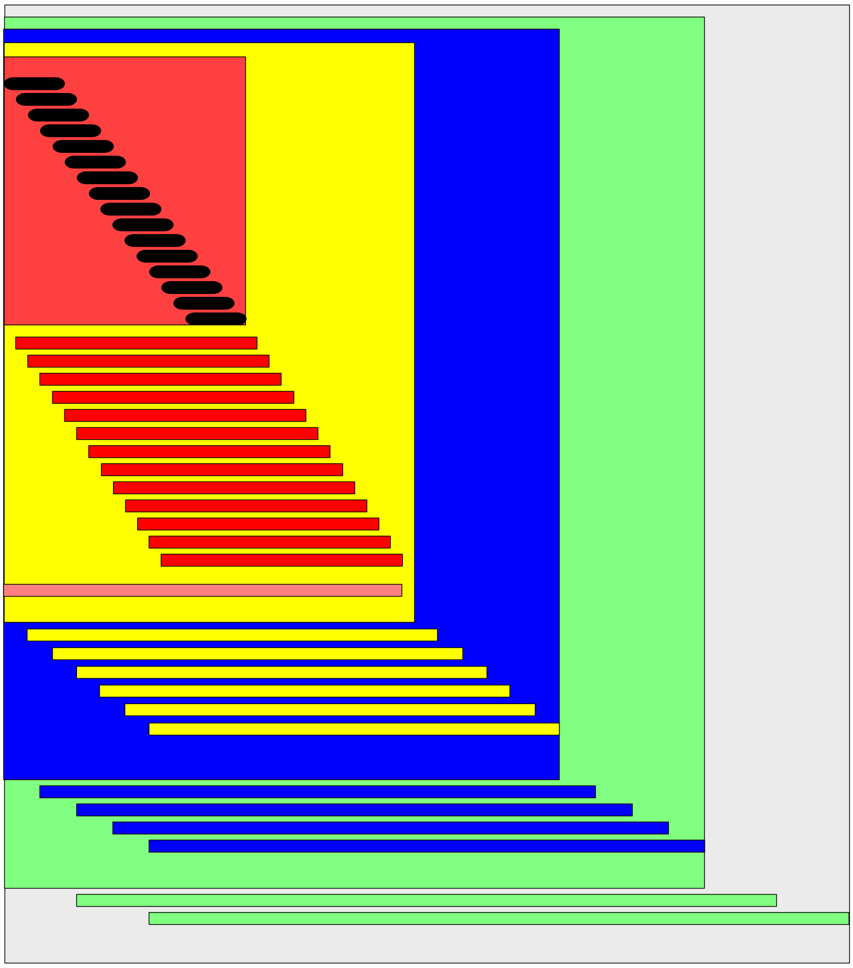

Figure 4 Cascades of cascades.

Proof. We have $\delta^1_j = \sum_{i=0}^{t-1} \delta_j[p-i]$. For a given j, $\delta_j[p-i] = 0$ except for the following values of i such that $0 \leq i \leq t-1$: $i = p-j$ where $\delta_j[j] = 1$; $i = p-j-1$ where $\delta_j[j+1] = -4$; $i = p-j-2$ where $\delta_j[j+2] = 4$; $i = p-j-4$ where $\delta_j[j+4] = -1$ (in the table it corresponds to the non zero values in a column, whose number is at most 4). Therefore, for $j > p$: $\delta^1_j = 0$; $\delta^1_p = 1$; $\delta^1_{p-1} = -4+1 = -3$; $\delta^1_{p-2} = 4-4+1 = 1$; $\delta^1_{p-3} = 0+4-4+1 = 1$; for $p-4 \geq j \geq p-t+1$, $\delta^1_{p-j} = -1+0+4-4+1 = 0$; $\delta^1_{p-t} = -1+0+4-4 = -1$; $\delta^1_{p-t-1} = -1+0+4 = 3$; $\delta^1_{p-t-2} = -1+0 = -1$; $\delta^1_{p-t-3} = -1$ and, for $j \leq p-t-4$, $\delta^1_j = 0$.
$\diamond$

Validity of the cascades. We have to see when the cascades are valid, that is, to determine how many groups we need at the beginning. For the cascade $\overrightarrow{\delta}^1[p,t]$ we note that the coordinates of any subsequence of the cascade, *i.e.*, the coordinates of some $\overrightarrow{\delta}^1[p,r]$, are all at least -1 except δ^1_{p-1}: which is -4 when $r = 1$ and then -3. Therefore such a cascade is valid as soon as we have at least 4 groups of size $p-1$ and one group of each other size $p-i$ ($2 \leq i \leq t+3$). To deal in general with the validity of cascades let us now introduce the notion of *h-balanced sequence.*

▶ **Definition 16.** Let h be a positive integer and let $\overrightarrow{\Phi} = \sum_{j=1}^{s} \overrightarrow{\varphi}^j$ be a cascade consisting of s k-deviations. We call this cascade h-balanced if, for any $1 \leq i \leq s$, the sum of the i

	...0	p	p-1	p-2	p-3	p-4	p-5	p-6	p-7	p-8	p-9	p-10	0...
$\delta[p]$	...0	1	-4	4	0	-1	0	0	0	0	0	0	0...
$+\delta[p-1]$	...0	0	1	-4	4	0	-1	0	0	0	0	0	0...
$+\delta[p-2]$	...0	0	0	1	-4	4	0	-1	0	0	0	0	0...
$+\delta[p-3]$	...0	0	0	0	1	-4	4	0	-1	0	0	0	0...
$+\delta[p-4]$	...0	0	0	0	0	1	-4	4	0	-1	0	0	0...
$+\delta[p-5]$	...0	0	0	0	0	0	1	-4	4	0	-1	0	0...
$+\delta[p-6]$	...0	0	0	0	0	0	0	1	-4	4	0	-1	0...
$= \overrightarrow{\delta}^1[p,7]$	...0	1	-3	1	1	0	0	0	-1	3	-1	-1	0...

Table 2 Computation of $\delta^1[p,7]$.

...	δ^1_p	δ^1_{p-1}	δ^1_{p-2}	δ^1_{p-3}	...	δ^1_{p-t}	δ^1_{p-t-1}	δ^1_{p-t-2}	δ^1_{p-t-3}	...
...0	1	-3	1	1	0...0	-1	3	-1	-1	0...

Table 3 Difference vector $\delta^1[p,t]$.

first vectors, namely $\sum_{j=1}^{i} \overrightarrow{\varphi}^j$, has all its coordinates greater than or equal to $-h$.

For example, the cascade $\overrightarrow{\delta}^1[p,t]$ described before is 4-balanced. The interest of this notion lies in the following fact: Let $p_{\max}$ be the largest index j that satisfies $\overrightarrow{\Phi}_j \neq 0$. Then, if we start from a partition with at least h groups of each size j, for $1 \leq j \leq p_{\max}$, an h-balanced sequence is valid.

Note that a sequence is itself composed of sub-sequences and the following lemma will be useful to bound the value h of a sequence.

▶ **Lemma 17.** *Let $\overrightarrow{\Phi}^1$ be an h_1-balanced sequence and $\overrightarrow{\Phi}^2$ be an h_2-balanced sequence. Then, $\overrightarrow{\Phi}^1 + \overrightarrow{\Phi}^2$ is a (max $\{h_1, h_2 - min_i \Phi^1_i\}$)-balanced sequence.*

Proof. As $\overrightarrow{\Phi}^1$ is h_1-balanced, the coordinates of any subsequence of $\overrightarrow{\Phi}^1$ are greater than or equal to $-h_1$. Consider a subsequence $\overrightarrow{\Phi}^1 + \overrightarrow{\Phi}^3$ where $\overrightarrow{\Phi}^3$ is a subsequence of $\overrightarrow{\Phi}^2$. The j-th coordinate is $\Phi^1_j + \Phi^3_j$; by definition $\Phi^3_j \geq -h_2$ and so $\Phi^1_j + \Phi^3_j \geq \Phi^1_j - h_2 \geq min_i \Phi^1_i - h_2$. ◇

The cascade $\overrightarrow{\delta}^2[p,t-2]$: We do now the following sequence of $t-2$ cascades $\overrightarrow{\delta}^2[p,t-2] = \sum_{i=0}^{t-3} \overrightarrow{\delta}^1[p-i,t]$. Altogether we have a sequence of $t(t-2)$ 4-deviations. There are a lot of cancellations and in fact, as shown in Claim 18, $\overrightarrow{\delta}^2[p,t-2]$ has only 10 non zero coordinates. Table 4 describes an example of computation of $\overrightarrow{\delta}^2[p,t-2]$ with $t=7$.

	...	p	p-1				p-5				p-9				p-13	p-14	...
$\delta^1[p,7]$	...0	1	-3	1	1	0	0	0	-1	3	-1	-1	0	...			
$+\delta^1$[p-1,7]	...0	0	1	-3	1	1	0	0	0	-1	3	-1	-1	0	...		
$+\delta^1$[p-2,7]	...0	0	0	1	-3	1	1	0	0	0	-1	3	-1	-1	0	...	
$+\delta^1$[p-3,7]	...0	0	0	0	1	-3	1	1	0	0	0	-1	3	-1	-1	0	...
$+\delta^1$[p-4,7]	...0	0	0	0	0	1	-3	1	1	0	0	0	-1	3	-1	-1	0...
$= \overrightarrow{\delta}^2[p,5]$	...0	1	-2	-1	0	0	-1	2	0	2	1	0	0	1	-2	-1	0...

Table 4 Computation of $\delta^2[p,5]$.

▶ Claim 18. For $3 \le t \le \frac{p}{2}$, the coordinates of the cascade $\overrightarrow{\delta}^2[p,t-2] = \sum_{i=0}^{t-3} \overrightarrow{\delta}^1[p-i,t]$ satisfy: $\delta^2_p = 1$, $\delta^2_{p-1} = -2$, $\delta^2_{p-2} = -1$, $\delta^2_{p-t+2} = -1$, $\delta^2_{p-t+1} = 2$, $\delta^2_{p-t-1} = 2$, $\delta^2_{p-t-2} = 1$, $\delta^2_{p-2t+2} = 1$, $\delta^2_{p-2t+1} = -2$, $\delta^2_{p-2t} = -1$, and $\delta^2_j = 0$ for all the others j (see Table 5). Furthermore this cascade is 4-balanced.

...	δ^2_p	δ^2_{p-1}	δ^2_{p-2}	...	δ^2_{p-t+2}	δ^2_{p-t+1}	δ^2_{p-t}	δ^2_{p-t-1}	δ^2_{p-t-2}	...	δ^2_{p-2t+2}	δ^2_{p-2t+1}	δ^2_{p-2t}	...
0	1	-2	-1	0	-1	2	0	2	1	0	1	-2	-1	0

Table 5 Difference vector $\delta^2[p,t-2]$.

Proof. We have $\delta^2_j = \sum_{i=0}^{t-3} \delta^1_j[p-i,t]$. Using the values of $\delta^1_j[p-i,t]$ given in Claim 23, we get that: for $j > p$, $\delta^2_j = 0$; $\delta^2_p = 1$; $\delta^2_{p-1} = -3+1 = -2$; $\delta^2_{p-2} = 1-3+1 = -1$; for $p-3 \ge j \ge p-t+3$, $\delta^2_j = 1+1-3+1 = 0$; $\delta^2_{p-t+2} = 1+1-3 = -1$; $\delta^2_{p-t+1} = 1+1 = 2$; $\delta^2_{p-t} = -1+1 = 0$; $\delta^2_{p-t-1} = 3-1 = 2$; $\delta^2_{p-t-2} = -1+3-1 = 1$; for $p-t-3 \ge j \ge p-2t+3$, $\delta^2_j = -1-1+3-1 = 0$; $\delta^2_{p-2t+2} = -1-1+3 = 1$; $\delta^2_{p-2t+1} = -1-1 = -2$, $\delta^2_{p-2t} = -1$, and for $j < p-2t$, $\delta^2_j = 0$.

Using Lemma 17 we get that $\overrightarrow{\delta}^2[p,t-2]$ is 7-balanced; but a careful analysis shows that this sequence is in fact 4-balanced. Indeed we will prove by induction that $\overrightarrow{\delta}^2[p,r] = \sum_{i=0}^{r-1} \overrightarrow{\delta}^1[p-i,t]$ is 4-balanced for any $r \le t-3$. That is true for $r=1$, as $\overrightarrow{\delta}^1[p,t]$ is 4-balanced. Suppose that it is true for r. We have $\overrightarrow{\delta}^2[p,r+1] = \overrightarrow{\delta}^2[p,r] + \overrightarrow{\delta}^1[p-r-1,t]$. All the coordinates of $\overrightarrow{\delta}^2[p,r]$ are by the computation above at least -3, and the coordinates of $\overrightarrow{\delta}^1[p-r-1,t]$ are greater than -1 except for $j = p-r-2$ where $\delta^1_{p-r-2}[p-r-1] = -4$; but $\delta^2_{p-r-2}[p,r] = 1$ (case $r=1$) or 2 (case $r>1$) and so all the coordinates of $\overrightarrow{\delta}^2[p,r+1]$ are at least -4. ◇

At this stage we could continue and do a cascade of $\overrightarrow{\delta}^2[p,t-2]$ but there is no more the phenomenon of cancellation. In fact we will use the following "symmetrization" trick. We will transform the cascade $\overrightarrow{\delta}^2[p,t-2]$ into a sequence $\overrightarrow{\zeta}^2[p]$ by doing some sequence of 1-deviations whose coordinates are given in Claim 19 The sequence obtained has only 8 non zero coefficients (4 with values 1 and 4 with values -1) arranged in a very symmetric nice way (that we will call *Nice Property*). Furthermore we will be able to iterate a cascade process on it many times keeping the property.

For $p \ge q+2$, we will denote by $\overrightarrow{\alpha}[p,q]$ the 1-deviation, where a vertex leaves a group of size $q+1$ for a group of size $p-1$ (valid as $p \ge q+2$). Let $\overrightarrow{\alpha}^1[p,q,r] = \sum_{i=0}^{r-1} \overrightarrow{\alpha}[p-i,q+i]$ denote a cascade of r such 1-deviations (we need $p-r+1 \ge q+r+2$ in order it is valid). The coordinates of $\overrightarrow{\alpha}^1[p,q,r]$ are given in the following Claim 19.

▶ Claim 19. For $p-r \ge q+r+1$, $\overrightarrow{\alpha}^1[p,q,r] = \sum_{i=0}^{r-1} \overrightarrow{\alpha}[p-i,q+i]$ has only 4 non zero coordinates namely $\alpha^1_p = 1$, $\alpha^1_{p-r} = -1$, $\alpha^1_{q+r} = -1$, and $\alpha^1_q = 1$.

▶ Claim 20. For $3 \le t \le \frac{p+1}{2}$, the coordinates of the sequence $\overrightarrow{\zeta}^2[p] = \overrightarrow{\delta}^2[p,t-2] + \overrightarrow{\alpha}^1[p-1,p-2t-1,t-2] + \overrightarrow{\alpha}^1[p-1,p-2t,2] + \overrightarrow{\alpha}^1[p-t+2,p-2t+1,1] + \overrightarrow{\alpha}^1[p-t,p-t-3,1]$ satisfy: $\zeta^2_p = 1$, $\zeta^2_{p-2} = -1$, $\zeta^2_{p-3} = -1$, $\zeta^2_{p-t} = 1$, $\zeta^2_{p-t-1} = 1$, $\zeta^2_{p-2t+2} = -1$, $\zeta^2_{p-2t+1} = -1$, $\zeta^2_{p-2t-1} = 1$ (see Table 6). Furthermore this cascade is still 4-balanced.

Proof. By Claim 19, we have the following coordinates:

- for $\overrightarrow{\alpha}^1[p-1,p-2t-1,t-2]$, $\alpha^1_{p-1} = 1$, $\alpha^1_{p-t+1} = -1$, $\alpha^1_{p-t-3} = -1$, $\alpha^1_{p-2t-1} = 1$;
- for $\overrightarrow{\alpha}^1[p-1,p-2t,2]$, $\alpha^1_{p-1} = 1$, $\alpha^1_{p-3} = -1$, $\alpha^1_{p-2t+2} = -1$, $\alpha^1_{p-2t} = 1$;
- for $\overrightarrow{\alpha}^1[p-t+2,p-2t+1,1]$, $\alpha^1_{p-t+2} = 1$, $\alpha^1_{p-t+1} = -1$, $\alpha^1_{p-2t+2} = -1$, $\alpha^1_{p-2t+1} = 1$;
- for $\overrightarrow{\alpha}^1[p-t,p-t-3,1]$, $\alpha^1_{p-t} = 1$, $\alpha^1_{p-t-1} = -1$, $\alpha^1_{p-t-2} = -1$, $\alpha^1_{p-t-3} = 1$.

...	ζ^2_p	ζ^2_{p-1}	ζ^2_{p-2}	ζ^2_{p-3}	...	ζ^2_{p-t}	ζ^2_{p-t-1}	...	ζ^2_{p-2t+2}	ζ^2_{p-2t+1}	ζ^2_{p-2t}	ζ^2_{p-2t-1}	...
...0	1	0	-1	-1	0...0	1	1	0...0	-1	-1	0	1	0...

Table 6 Difference vector $\zeta^2[p]$.

Therefore, using these values and the values of the coordinates of δ^2_j given in claim 18, we get $\zeta^2_p = 1$, $\zeta^2_{p-1} = -2+1+1 = 0$, $\zeta^2_{p-2} = -1$, $\zeta^2_{p-3} = 0-1 = -1$, $\zeta^2_{p-t+2} = -1+1 = 0$, $\zeta^2_{p-t+1} = 2-1-1 = 0$, $\zeta^2_{p-t} = 0+1 = 1$, $\zeta^2_{p-t-1} = 2-1 = 1$, $\zeta^2_{p-t-2} = 1-1 = 0$, $\zeta^2_{p-t-3} = 0-1+1 = 0$, $\zeta^2_{p-2t+2} = 1-1-1 = -1$, $\zeta^2_{p-2t+1} = -2+1 = -1$, $\zeta^2_{p-2t} = -1+1 = 0$ $\zeta^2_{p-2t-1} = 0+1$.

To prove that $\overrightarrow{\zeta}^2[p]$ is 4-balanced, apply Lemma 17 with $\overrightarrow{\Phi}^1 = \overrightarrow{\delta}^2[p,t-2]$ and $\overrightarrow{\Phi}^2 = \overrightarrow{\alpha}^1[p-1,p-2t-1,t-2]+\overrightarrow{\alpha}^1[p-1,p-2t,1]+\overrightarrow{\alpha}^1[p-t+2,p-2t+1,1]+\overrightarrow{\alpha}^1[p-t,p-t-3,1]$. We have that $h_1 = 4$ and furthermore all the coefficients of $\overrightarrow{\Phi}^1$ are greater than -2 and $\overrightarrow{\Phi}^2$ is 2-balanced. Hence, $\overrightarrow{\zeta}^2[p]$ is $\max(4,2+2) = 4$-balanced. ◇

Table 7 shows an example with $t = 7$.

	...	p	p-1			...			p-7	p-8			...				p-15	...
$\delta^2[p,5]$	0	1	-2	-1	0	0	-1	2	0	2	1	0	0	1	-2	-1	0	
$+\alpha$[p-1,p-15,5]	0	0	1	0	0	0	0	-1	0	0	0	-1	0	0	0	0	1	0
$+\alpha$[p-1,p-14,2]	0	0	1	0	-1	0	0	0	0	0	0	-1	0	-1	0	1	0	0
$+\alpha$[p-5,p-13,1]	0	0	0	0	0	0	1	-1	0	0	0	0	0	-1	1	0	0	0
$+\alpha$[p-7,p-10,1]	0	0	0	0	0	0	0	0	1	-1	-1	1	0	0	0	0	0	0
$= \overrightarrow{\zeta}^2[p]$	0	1	0	-1	-1	0	0	0	1	1	0	0	0	-1	-1	0	1	0

Table 7 Computation of $\zeta^2[p]$ with $t = 7$.

▶ **Definition 21. Nice Property**: Let $k \geq 2$ be a positive integer. We will say the sequence $\overrightarrow{\zeta}^k[p]$ has the *Nice Property*, if there exist 3 integers $a(k)$, $b(k)$, and $s(k)$ satisfying $1 < a(k) < b(k) < 2a(k)$ and $b(k) < s(k) - 1 < p/2$ and such that all coordinates of $\overrightarrow{\zeta}^k$ are null except for:

- $\zeta^k_p = \zeta^k_{p+1-2s(k)} = 1$,
- $\zeta^k_{p-a(k)} = \zeta^k_{p-b(k)} = \zeta^k_{p+1-2s(k)+b(k)} = \zeta^k_{p+1-2s(k)+a(k)} = -1$, and
- $\zeta^k_{p+1-s(k)} = \zeta^k_{p-s(k)} = 1$.

We note the symmetry of the coordinates, as for any j, $\zeta^k_{p-j} = \zeta^k_{p+1-2s(k)+j}$. As an example, the sequence $\overrightarrow{\zeta}^2[p]$ satisfies the Nice Property with $a(2) = 2$, $b(2) = 3$ and $s(2) = t+1$ and is 4-balanced. Now we will show how starting with a sequence $\overrightarrow{\zeta}^k[p]$ satisfying the Nice Property we can construct a sequence $\overrightarrow{\zeta}^{k+1}[p]$ having still the Nice Property.

▶ Claim 22. **Main construction**: Let $\overrightarrow{\zeta}^k[p]$ be a sequence satisfying the Nice Property with parameters $a(k), b(k), s(k)$. Then, we can construct a sequence $\overrightarrow{\zeta}^{k+1}[p]$ satisfying the following properties:

- $\overrightarrow{\zeta}^{k+1}[p]$ satisfies the Nice Property with parameters
  - $a(k+1) = b(k)$,
  - $b(k+1) = b(k) + a(k)$,

- $s(k+1) = s(k) + a(k)r(k)/2$, where $r(k)$ is the greatest even integer such that $r(k)a(k) + b(k) < s(k) - 1$;

- if $\overrightarrow{\zeta}^k[p]$ is $h(k)$-balanced, then $\overrightarrow{\zeta}^{k+1}[p]$ is $(h(k)+1)$-balanced;
- $\overrightarrow{\zeta}^{k+1}[p]$ contains $r(k)+1$ sequences $\overrightarrow{\zeta}^k[p]$.

Proof. We will first do a cascade of $\overrightarrow{\zeta}^k[p]$, but we will take values of the parameters differing by a multiple of $a(k)$ in order for some of the coordinates to cancel. More precisely let us define $\overrightarrow{\Psi}^r = \sum_{j=0}^{r} \overrightarrow{\zeta}^k[p - ja(k)]$. Using the values of Definition 21, we get the following values for the non zero coordinates:

(1) $\psi^r_p = 1$; $\psi^r_{p-ja(k)} = -1+1 = 0$ for $0 < j \leq r$ (cancellation phenomenom); $\psi^r_{p-(r+1)a(k)} = -1$;

(2) $\psi^r_{p+1-2s(k)+a(k)} = -1$; $\psi^r_{p+1-2s(k)-(j-1)a(k)} = 1-1 = 0$ for $0 < j \leq r$ (cancellation); $\psi^r_{p+1-2s(k)-ra(k)} = 1$;

(3) for $0 \leq j \leq r$, $\psi^r_{p-b(k)-ja(k)} = -1$;

(4) for $0 \leq j \leq r$, $\psi^r_{p+1-2s(k)+b(k)-ja(k)} = -1$;

(5) for $0 \leq j \leq r$, $\psi^r_{p+1-s(k)-ja(k)} = \psi^r_{p-s(k)-ja(k)} = 1$.

All the indices of the coordinates are different as $a(k) < b(k) < 2a(k)$ and as soon as we choose r even and nonzero such that $p - b(k) - ra(k) > p + 1 - s(k)$ (that is equivalent to $ra(k) + b(k) < s(k) - 1$). Let us denote $a(k+1) = b(k)$, $b(k+1) = b(k) + a(k)$ and $s(k+1) = s(k) + a(k)r/2$. Then $\overrightarrow{\Psi}^r$ has already part of the Nice Property for $k+1$. Indeed we have:

- $\psi^r_p = 1$ by (1) and $\psi^r_{p+1-2s(k+1)} = 1$ by (2) with $j = r$ (as $2s(k+1) = 2s(k) + ra(k)$);
- $\psi^r_{p-a(k+1)} = \psi^r_{p-b(k)} = -1$, $\psi^r_{p-b(k+1)} = \psi^r_{p-b(k)-a(k)} = -1$ by (3) with $j = 0, 1$;
- $\psi^r_{p+1-2s(k+1)+b(k+1)} = -1$, $\psi^r_{p+1-2s(k+1)+a(k+1)} = -1$ by (4) with $j = r-1, r$;
- $\psi^r_{p+1-s(k+1)} = \psi^r_{p-s(k+1)} = 1$ by (5) with $j = r/2$.

The remaining non zero coordinates are in number $4r$: firstly there are r values -1, namely $\psi^r_{p-b(k)-ja(k)} = -1$, for $2 \leq j \leq r$, and $\psi^r_{p-(r+1)a(k)} = -1$; then there are $2r$ values 1, namely $\psi^r_{p+1-s(k+1)} = \psi^r_{p-s(k+1)} = 1$, for $j \neq r/2$ and finally r values -1, namely $\psi^r_{p+1-2s(k)+a(k)} = -1$ and $\psi^r_{p+1-2s(k)+b(k)-ja(k)} = -1$, for $0 \leq j \leq r-2$. These values are disposed in a very symmetric way and can be written: for the values -1, in the form $\psi^r_{p-x_m}$ and $\psi^r_{p+1-2s(k+1)+x_m}$; and for the values 1, in the form $\psi^r_{p-y_m}$ and $\psi^r_{p+1-2s(k+1)+y_m}$ with $x_m < y_m$ $(0 \leq m \leq r-1)$. Furthermore, these r quadruples of values can be canceled by adding to $\overrightarrow{\Psi}^r$ the r sequences $\overrightarrow{\alpha}^1[p - x_m, p+1-2s(k+1)+x_m, y_m - x_m]$.

We claim that the sequence so obtained, with partition vector $\overrightarrow{\Psi}^r + \sum_{m=0}^{r-1} \overrightarrow{\alpha}^1[p-x_m, p+1-2s(k+1)+x_m, y_m-x_m]$, satisfies the Nice Property with parameters $a(k+1), b(k+1)$ and $s(k+1)$. Indeed $a(k+1) = b(k) < b(k) + a(k) = b(k+1)$, $b(k+1) = b(k) + a(k) < b(k) + b(k) = 2a(k+1)$ and $b(k+1) = b(k) + a(k) < s(k) - 1 + a(k) \leq s(k+1) - 1$ as $r \geq 2$. We also have to ensure in the computations that p is chosen so that $p \geq 2s(k) - 1$. In order to get the maximum number of deviations we will consider this sequence for the largest possible even integer r satisfying $ra(k) + b(k) < s(k) - 1$, denoted $r(k)$ and we will denote the sequence for this $r(k)$ by $\overrightarrow{\zeta}^{k+1}[p]$.

We now prove that $\overrightarrow{\zeta}^{k+1}[p]$ is $h(k)+1$-balanced. We first prove by induction that $\overrightarrow{\Psi}^r$ is $(h(k)+1)$-balanced. That is true for $r = 0$ as $\overrightarrow{\zeta}^k[p]$ is $h(k)$-balanced. Then suppose it is true for some r ; we apply Lemma 17 with $\overrightarrow{\Phi}^1 = \overrightarrow{\Psi}^r$ and $\overrightarrow{\Phi}^2 = \overrightarrow{\zeta}^k[p-(r+1)a(k)]$. We have that $h_1 = h(k)+1$ by induction hypothesis and furthermore all the coefficients of $\overrightarrow{\Phi}^1$ are greater than -1; furthermore $\overrightarrow{\Phi}^2$ is $h(k)$-balanced and so $\overrightarrow{\Psi}^{r+1}$ is $(\max(h(k)+1, h(k)+1) =$

$h(k)+1)$-balanced. Then when we add an $\overrightarrow{\alpha}^1[p-x_m, p+1-2s(k+1)+x_m, y_m-x_m]$ which is 1-balanced we still get an $(\max(h(k)+1, 1+1) = h(k)+1$-balanced sequence.

Finally, by construction, we get that $\overrightarrow{\zeta}^{k+1}[p]$ contains $r(k)+1$ sequences $\overrightarrow{\zeta}^k[p]$. $\diamond$

End of the proof of Theorem 14.

At this stage we have built a sequence $\overrightarrow{\zeta}^2[p]$ which satisfies the Nice Property with $a(2)=2$, $b(2)=3$ and $s(2)=t+1$ and is $h(2)$=4-balanced. Furthermore, it contains $t(t-2)$ 4-deviations. See Claim 20.Then, for some well-chosen K (to be defined later) we can apply $K-2$ times the main construction (Claim 22) to construct a sequence $\overrightarrow{\zeta}^K[p]$ which satisfies the Nice Property with parameters $a(K)$, $b(K)$ and $s(K)$ and is $h(K)$-balanced.

We have $a(k) = b(k-1)$, $b(k) = b(k-1)+a(k-1) = b(k-1)+b(k-2)$ and we recognize the Fibonacci recurrence relation. The k^{th} Fibonacci number $F(k)$ is denoted as follows:

$$F(k) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right).$$

Then, as $a(2) = 2 = F(3)$ and $b(2) = 3 = F(4)$, we get $a(K) = F(K+1)$ and $b(K) = F(K+2)$. In fact in what follows we will use only that $a(K) \leq 2^{K-1}$ and $b(K) \leq 2^K$. We have $s(k+1) = s(k) + a(k)r(k)/2$; but $a(k)r(k) < s(k-1) - b(k) < s(k-1)$ and so $s(k+1) < (3/2) \times s(k)$ and $s(K) < s(2)(3/2)^{K-2} = (t+1)(3/2)^{K-2}$.

Recall that we should have $p \geq 2s(K)-1$ so we choose $p = 2s(K)$. Furthermore by induction we have that $h(K) = K+2$. So we need to start with a partition containing at least $K+2$ groups of each size i, $1 \leq i \leq p$. It is easy to obtain such a starting partition from the initial partition — which consists of n groups of size 1 — by doing a sequence of 1-deviation of size $(K-2)p(p+1)/2$; indeed we can create a group of any size i with $(i-1)$ 1-deviations. Therefore, we will take $n = (K-2)p(p+1)/2 \leq (K-2)s(K)(2s(K)+1)$. Using the inequality $s(K) < (t+1)(3/2)^{K-2}$ we get that

$$n = \mathcal{O}(t^2 K (3/2)^{2K}). \tag{1}$$

On the other hand we have to lower bound the number of deviations. By construction $\overrightarrow{\zeta}^{k+1}[p]$ contains $r(k)+1$ sequences $\overrightarrow{\zeta}^k[p]$ and so, contains $t(t-2)\prod_{k=2}^{K-1}(r(k)+1)$ 4-deviations, as $\overrightarrow{\zeta}^2[p]$ contains $t(t-2)$ 4-deviations. Recall that $r(k)$ is the greatest even integer r such that $ra(k)+b(k) < s(k)-1$ and so $r(k) \geq \lfloor \frac{s(k)-1-b(k)}{a(k)} \rfloor - 1$. Using the fact that $b(k)+1 \leq 2a(k)$ and $s(k) > s(2)-1 = t$, and $a(k) \leq a(K) < 2^{K-1}$ we get $r(k) \geq \frac{t}{2^{K-1}} - 3$. Then $\prod_{k=2}^{K-1}(r(k)+1) \geq (\frac{t}{2^{K-1}} - 2)^{K-2}$ and the number D of deviations satisfies:

$$D = \Omega(t^2(\frac{t}{2^{K-1}} - 2)^{K-2}). \tag{2}$$

We have now to choose K as a function of t. In order for the number of deviations as given by Equation 2 to increase we need that 2^{K-1} is small compared to t that is $K << \log_2(t)$. However in view of Equation 1 we want to choose K the largest possible. Therefore, a good choice is $K = 1/2(\log_2(t))$.

In that case, we get by Equation 1 that $n = \mathcal{O}(t^2 \log_2(t)(3/2)^{\log_2(t)})$, or equivalently $\log_2(n) = \mathcal{O}(2\log_2(t) + \log_2(\log_2(t) + \log_2(t)(\log_2(3) - \log_2(2)))$. Using $\log_2(3) - \log_2(2) > 0.585$ and the fact that for t large enough $\log_2(\log_2(t)) < 0.014\log_2(t)$ we get $\log_2(n) = \mathcal{O}(2.6\log_2(t))$, that is $n = \mathcal{O}(t^{2.6})$. On the other hand we get by Equation 2: $D = \Omega((t^{1/2})^{1/2\log_2(t)}) = \Omega(t^{1/4\log_2(t)})$ and so $D = \Omega(n^{clog_2(n)})$ with $c = \frac{1}{4\times(2.6)^2} \simeq 1/27$, thereby proving Theorem 14.

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5.2 Appendix: Proof for the case k = 3 (Theorem 13)

The proof uses the same idea that for $k = 4$ but is simpler as we can only do a limited iteration of cascades, which use a lot of 3-deviations $\gamma[p]$ (the only 3-deviation which does not increase the global utility).

Definition of $\gamma[p]$: Consider a partition P containing at least three groups of size $p-1$ and a group of size $p-3$. In the 3-deviation $\gamma[p]$ one vertex in each of the 3 groups of size $p-1$ moves to the group of size $p-3$ to form a new group of size p. The example given at the beginning of section 5 corresponds to the case $p = 4$. So, after this 3-deviation we get a new partition P' with one more group of size p, 3 less groups of size $p-1$, 3 more groups of size $p-2$ and one less group of size $p-3$. This is expressed by the coordinates of the associated difference vector (where we omit the bracket $[p]$ for ease of reading). Note that such a deviation is valid only if there are 3 groups of size $p-1$ and one group of size $p-3$.

Difference vector $\overrightarrow{\gamma}[p]$: The difference vector $\overrightarrow{\gamma}[p]$ has the following coordinates: $\gamma_p = 1, \gamma_{p-1} = -3, \gamma_{p-2} = 3, \gamma_{p-3} = -1$ and $\gamma_j = 0$ for all other values of j. See Table 8.

...	γ_p	γ_{p-1}	γ_{p-2}	γ_{p-3}	...
...0	1	-3	3	1	0...

Table 8 Difference vector of $\gamma[p]$.

The cascade $\overrightarrow{\gamma}^1[p,t]$: Like for $k = 4$, we do a cascade consisting of of t 3-deviations $\gamma[p], \gamma[p-1], \gamma[p-2], \dots, \gamma[p-t+1]$. We will denote this cascade by its difference vector $\overrightarrow{\gamma}^1[p,t] = \sum_{i=0}^{t-1} \overrightarrow{\gamma}[p-i]$. We will determine the coordinates of $\overrightarrow{\gamma}^1[p,t]$ in Claim 23, but first let us see how it works on a basic example.

p	13	12	11	10	9	8	7	6	5	4	3	2	1
$\overrightarrow{\gamma[13]}$	1	-3	3	-1									
$\overrightarrow{\gamma[12]}$		1	-3	3	-1								
$\overrightarrow{\gamma[11]}$			1	-3	3	-1							
$\overrightarrow{\gamma[10]}$				1	-3	3	-1						
$\overrightarrow{\gamma[9]}$					1	-3	3	-1					
$\overrightarrow{\gamma[8]}$						1	-3	3	-1				
$\gamma^1[13,6]$	1	-2	1	0	0	0	-1	2	-1	0	0	0	0

Table 9 Computation of $\gamma^1[13,6]$.

Basic example: We consider the partition P^0 consisting of $g \geq 3$ groups of each size i for $1 \leq i \leq 12$. The partition vector associated $\overrightarrow{\Lambda}(P^0)$ satisfies $\lambda_i^0 = g$ for $1 \leq i \leq 12$ and $\lambda_j^0 = 0$ for $j \geq 13$. In the example of Table 9, we choose $t = 6$ and so we do sucessively $\overrightarrow{\gamma}[13-i]$. The non zero coordinates of $\overrightarrow{\gamma}[13-i]$ are indicated in the corresponding line. The jth coordinate of $\overrightarrow{\gamma}^1[13,r]$ is obtainedby summing the numbers of the first r lines and column j. The coordinate indicates the number of groups of size j created (if positive) or deleted (if negative). If we look at the cascade obtained after 4 3-deviations we note a remarkable phenomenon of cancellation as the coordinate γ_{10}^1 of this cascade equals 0. That comes from the fact that we have sucessively deleted a group of size 10 with $\overrightarrow{\gamma}[13]$, then created 3 new groups with $\overrightarrow{\gamma}[12]$, then deleted 3 groups with $\overrightarrow{\gamma}[11]$ and finally created a new group with $\overrightarrow{\gamma}[10]$. This cancellation stays for all the other deviations. Similarly, the coordinate γ_9^1 of the cascade $\overrightarrow{\gamma}^1[13,5]$ equals 0 and so on. In fact as indicated in the

Claim 23 the vector $\overrightarrow{\gamma}^1[p,t]$ has only 6 non zero coordinates.

▶ Claim 23. For $3 \leq t \leq p-3$, the coordinates of the cascade $\overrightarrow{\gamma}^1[p,t] = \sum_{i=0}^{t-1} \overrightarrow{\gamma}[p-i]$ satisfy: $\gamma_p^1 = 1$, $\gamma_{p-1}^1 = -2$, $\gamma_{p-2}^1 = 1$, $\gamma_{p-t}^1 = -1$, $\gamma_{p-t-1}^1 = 2$, $\gamma_{p-t-2}^1 = -1$ and $\gamma_j^1 = 0$ for all the others j. Furthermore, this cascade is 3-balanced.

...	γ_p^1	γ_{p-1}^1	γ_{p-2}^1		...	γ_{p-t}^1	γ_{p-t-1}^1	γ_{p-t-2}^1	...
...0	1	-2	1	0...0	-1	2	-1	0...	

Table 10 Difference vector $\gamma^1[p,t]$.

Proof. We have $\gamma_j^1 = \sum_{i=0}^{t-1} \gamma_j[p-i]$. For a given j, $\gamma_j[p-i] = 0$ except for the following values of i such that $0 \leq i \leq t-1$: $i = p-j$ where $\gamma_j[j] = 1$; $i = p-j-1$ where $\gamma_j[j+1] = -3$; $i = p-j-2$ where $\gamma_j[j+2] = 3$; $i = p-j-3$ where $\gamma_j[j+3] = -1$ (in the table it corresponds to the consecutive non zero values in a column which are at most 4). Therefore, for $j > p$: $\gamma_j^1 = 0$; $\gamma_p^1 = 1$; $\gamma_{p-1}^1 = -3+1 = -2$; $\gamma_{p-2}^1 = 3-3+1 = 1$; for $p-3 \geq j \geq p-t+1$, $\gamma_{p-j}^1 = -1+3-3+1 = 0$; $\gamma_{p-t}^1 = -1+3-3 = -1$; $\gamma_{p-t-1}^1 = -1+3 = 2$; $\gamma_{p-t-2}^1 = -1$ and, for $j < p-t-2$, $\gamma_j^1 = 0$. Finally we note that the coordinates of any subsequence of the cascade, *i.e.* the coordinates of $\overrightarrow{\gamma}^1[p,r]$, are all at least -1 except γ_{p-1}^1: which is -3 when $r = 1$ and then -2, thereby proving that the cascade is 3-balanced. ◇

In our example we can do the cascade till $t = 10$ and in general we can do it till $t = p-3$. However, it is better to choose a t smaller (we will see after that a good value is such that $p - 3t > 0$) and repeat the previous cascade but shifted, and so, to do a cascade of cascades. More precisely we do now the following sequence of $t-1$ cascades $\overrightarrow{\gamma}^2[p,t-1] = \sum_{i=0}^{t-2} \overrightarrow{\gamma}^1[p-i,t]$. Altogether we have a sequence of $t(t-1)$ 3-deviations.

p	13	12	11	10	9	8	7	6	5	4	3	2	1
$\gamma^1[13,4]$	1	-2	1	0	-1	2	-1						
$\gamma^1[12,4]$		1	-2	1	0	-1	2	-1					
$\gamma^1[11,4]$			1	-2	1	0	-1	2	-1				
$\gamma^2[13,3]$	1	-1	0	-1	0	1	0	1	-1				
$\gamma^2[12,3]$		1	-1	0	-1	0	1	0	1	-1			
$\gamma^3[13,2]$	1	0	-1	-1	-1	1	1	1	0	-1			
$\alpha^1[11,4,3]$	0	0	1	0	0	-1	-1	0	0	1			
$\zeta^3[13,2]$	1	0	0	-1	-1	0	0	1	0	0			
$\zeta^3[12,2]$		1	0	0	-1	-1	0	0	1	0	0		
$\zeta^3[11,2]$			1	0	0	-1	-1	0	0	1	0	0	
$\zeta^3[10,2]$				1	0	0	-1	-1	0	0	1	0	0
$\zeta^4[13,4]$	1	1	1	0	-2	-2	-2	0	1	1	1	0	0

Table 11 Example of cascade of cascades.

In the example (see Table 11), we choose $t = 4$ and so after having done $\overrightarrow{\gamma}^1[13,4]$ we do $\overrightarrow{\gamma}^1[12,4]$ and $\overrightarrow{\gamma}^1[11,4]$. We can see again a phenomenon of cancellation as the coordinate γ_{11}^2 of this cascade of cascade equals 0. That comes from the fact that we have successively created a group of size 11 with $\overrightarrow{\gamma}^1[13,4]$, then deleted 2 groups with $\overrightarrow{\gamma}^1[12,4]$ and finally created a new group with $\overrightarrow{\gamma}^1[11,4]$. Similarly the coordinate γ_7^2 of this cascade of cascade equals 0. This cancellation stays for all the other deviations. In the general case there are a lot of cancellations and in fact, as shown in the next Claim 24, $\overrightarrow{\gamma}^2[p,t-1]$ has only 6 non zero coordinates.

▶ Claim 24. For $3 \leq t \leq \frac{p-1}{2}$, the coordinates of the cascade $\overrightarrow{\gamma}^2[p,t-1] = \sum_{i=0}^{t-2} \overrightarrow{\gamma}^1[p-i,t]$ satisfy: $\gamma_p^2 = 1$, $\gamma_{p-1}^2 = -1$, $\gamma_{p-t+1}^2 = -1$, $\gamma_{p-t-1}^2 = 1$, $\gamma_{p-2t+1}^2 = 1$, $\gamma_{p-2t}^2 = -1$ and $\gamma_j^2 = 0$ for all the others j. Furthermore this cascade is 3-balanced.

$\cdots$	γ_p^2	γ_{p-1}^2	$\cdots$	γ_{p-t+1}^2		γ_{p-t-1}^2	$\cdots$	γ_{p-2t+1}^2	γ_{p-2t}^2	$\cdots$
0	1	-1	0	-1	0	1	0	1	-1	0

Table 12 Difference vector $\delta^2[p,t-2]$.

Proof. We have $\gamma_j^2 = \sum_{i=0}^{t-2} \gamma_j^1[p-i,t]$. Using the values of $\gamma_j^1[p-i,t]$ given in Claim 23, we get that: for $j > p$, $\gamma_j^2 = 0$; $\gamma_p^2 = 1$; $\gamma_{p-1}^2 = 1-2 = -1$; for $p-2 \geq j \geq p-t+2$, $\gamma_j^2 = 1-2+1 = 0$; $\gamma_{p-t+1}^2 = 1-2 = -1$; $\gamma_{p-t}^2 = -1+0+1 = 0$; $\gamma_{p-t-1}^2 = 2-1 = 1$; for $p-t-2 \geq j \geq p-2t+2$, $\gamma_j^2 = -1+2-1 = 0$; $\gamma_{p-2t+1}^2 = -1+2 = 1$; $\gamma_{p-2t}^2 = -1$ and for $j < p-2t$, $\gamma_j^2 = 0$.

Here again we can see that after any number r of 3-deviations the coordinates of the sequence are always at least greater than or equal to -3. Indeed the -3 appears only after the first 3-deviation for groups of size $p-1$; otherwise, when a -3 appears it is after a $+3$ and so all the other coordinates are in fact at least -2. Thus the cascade is 3-balanced. ◇

We can now repeat the cascade of cascades $\overrightarrow{\gamma}^2[p,t-1]$ but shifted. More precisely we do now the following sequence of $t-2$ cascades $\overrightarrow{\gamma}^3[p,t-2] = \sum_{i=0}^{t-3} \overrightarrow{\gamma}^2[p-i,t-1]$. Altogether we have a sequence of $t(t-1)(t-2)$ 3-deviations. In the example (see Table 11), after having done $\overrightarrow{\gamma}^2[13,3]$ we do $\overrightarrow{\gamma}^2[12,3]$. We can see again a phenomenon of cancellation as the coordinate γ_{12}^3 of this cascade of cascades equals 0. That comes from the fact that we have successively deleted a group of size 12 after $\overrightarrow{\gamma}^2[13,3]$, then created a new group after $\overrightarrow{\gamma}^2[12,3]$. Similarly the coordinate γ_5^3 of this cascade of cascade equals 0. Here we have also deleted one group of size 5, then created one such group. This cancellation stays for all the other deviations. In the general case there are a lot of cancellations and in fact as shown in the next Claim 25, $\overrightarrow{\gamma}^2[p,t-1]$ has only 8 non zero coordinates.

▶ Claim 25. For $3 \leq t \leq \frac{p+2}{3}$, the coordinates of the cascade $\overrightarrow{\gamma}^3[p,t-2] = \sum_{i=0}^{t-3} \overrightarrow{\gamma}^2[p-i,t-1]$ satisfy: $\gamma_p^3 = 1$, $\gamma_{p-t+2}^3 = \gamma_{p-t+1}^3 = \gamma_{p-t}^3 = -1$, $\gamma_{p-2t+3}^3 = \gamma_{p-2t+2}^3 = \gamma_{p-2t+1}^3 = 1$, $\gamma_{p-3t+3}^3 = -1$, and $\gamma_j^3 = 0$ for all the others j. Furthermore this cascade is 4-balanced

$\cdots$	γ_p^3	$\cdots$	γ_{p-t+2}^3	γ_{p-t+1}^3	γ_{p-t}^3	$\cdots$	γ_{p-2t+3}^2	γ_{p-2t+2}^2	γ_{p-2t+1}^2	$\cdots$	γ_{p-3t+3}^2	$\cdots$
0	1	0	-1	-1	-1	0	1	1	1	0	-1	0

Table 13 Difference vector $\gamma^3[p,t-2]$.

Proof. We have $\gamma_j^3 = \sum_{i=0}^{t-3} \gamma_j^2[p-i,t-1]$. Using the values of $\gamma_j^2[p-i,t-1]$ given in Claim 24, we get that: for $j > p$, $\gamma_j^2 = 0$; $\gamma_p^3 = 1$; for $p-1 \geq j \geq p-t+3$, $\gamma_j^3 = -1+1 = 0$; $\gamma_{p-t+2}^3 = 0-1 = -1$; $\gamma_{p-t+1}^3 = -1+0 = -1$; $\gamma_{p-t}^3 = 0-1 = -1$; for $p-t-1 \geq j \geq p-2t+4$, $\gamma_j^3 = 1+0-1 = 0$; $\gamma_{p-2t+3}^3 = 1+0 = 1$; $\gamma_{p-2t+2}^3 = 0+1 = 1$; $\gamma_{p-2t+1}^3 = 1+0 = 1$; for $p-2t \geq j \geq p-3t+4$, $\gamma_j^3 = -1+1 = 0$; $\gamma_{p-3t+3}^3 = 0-1 = -1$; and for $j < p-3t+3$, $\gamma_j^3 = 0$.

Here again a careful but tedious analysis of all the subsequence of deviations indicate that all their coordinates are at least -3. However using Lemma 17 we can easily prove that it is 4-balanced. In fact, we will prove by induction that $\sum_{i=0}^{r} \overrightarrow{\gamma}^2[p-i,t-1]$ is 4-balanced for any $r \leq t-3$. That is true for $r = 0$, as $\overrightarrow{\gamma}^2[p,t-1]$ is 3-balanced. Suppose that $\sum_{i=0}^{r} \overrightarrow{\gamma}^2[p-i,t-1]$ is 4-balanced for some $r \leq t-3$. We apply Lemma 17 with $\overrightarrow{\Phi}^1 = \sum_{i=0}^{r} \overrightarrow{\gamma}^2[p-i,t-1]$ and $\overrightarrow{\Phi}^2 = \overrightarrow{\gamma}^2[p-r-1,t-1]$. We have by induction hypothesis that $h_1 = 4$ and furthermore all the coefficients of $\overrightarrow{\Phi}^1$ are greater than -1 by Claim 24 when $r = 0$ or Claim 25 when $r > 0$ and so $\min_i \Phi_i^1 = -1$. Finally $\overrightarrow{\Phi}^2$ is 3-balanced and so $\sum_{i=0}^{r+1} \overrightarrow{\gamma}^2[p-i,t-1]$ is also $\max(4,3+1) = 4$-balanced and by induction $\overrightarrow{\gamma}^3[p,t-2]$ is 4-balanced. ◇

We can now again repeat t times the cascades $\overrightarrow{\gamma}^3[p,t-2]$ creating a sequence of cascades that we call $\overrightarrow{\gamma}^4[p,t] = \sum_{i=0}^{t-1} \overrightarrow{\gamma}^3[p-i,t-2]$. This is enough to prove Theorem 13. But we will use the trick used for $k = 4$ which consists in doing after $\overrightarrow{\gamma}^3[p,t-2]$ a sequence of 1-deviations which will reduce to 4 the number of non zero coordinates. This slightly improves the ratio between the number of 3-deviations and the number of vertices. So, after having done $\overrightarrow{\gamma}^3[p,t-2]$, we apply the cascade $\overrightarrow{\alpha}^1[p-t+2,p-3t+3,t-1]$ (see Claim 19). Let $\overrightarrow{\zeta}^3[p,t-2]$ be the sequence we obtain. As shown in Claim 26, it has only 4 non zero coordinates.

▶ Claim 26. For $3 \leq t \leq p/2$ the coordinates of $\overrightarrow{\zeta}^3[p,t-2] = \overrightarrow{\gamma}^3[p,t-2] + \overrightarrow{\alpha}^1[p-t+2,p-3t+3,t-1]$ satisfy: $\zeta_p^3 = 1, \zeta_{p-t+1}^3 = \zeta_{p-t}^3 = -1, \zeta_{p-2t+1}^3 = 1$ and $\zeta_j^3 = 0$ for all the others j. Furthermore it is 4-balanced.

Proof. Compared to $\overrightarrow{\gamma}^3[p,t-2]$ only 4 coordinates have been changed and we get $\zeta_{p-t+2}^3 = \gamma_{p-t+2}^3 + \alpha^1(p-t+2) = -1+1 = 0$ $\zeta_{p-2t+3}^3 = \zeta_{p-2t+2}^3 = 1-1 = 0$ and $\zeta_{p-3t+3}^3 = -1+1 = 0$. All the other coordinates remain the same. To prove that $\overrightarrow{\zeta}^3[p,t-2]$ is 4-balanced, we apply Lemma 17 with $\overrightarrow{\Phi}^1 = \overrightarrow{\gamma}^3[p,t-2]$ and $\overrightarrow{\Phi}^2 = \overrightarrow{\alpha}^1[p-t+2,p-3t+3,t-1]$. We have proved that $\overrightarrow{\Phi}^1$ is 4-balanced and by Claim 25 $\min_i \Phi_i^1 = -1$. Furthermore, $\overrightarrow{\Phi}^2$ is 1-balanced and so $\overrightarrow{\zeta}^3[p,t-2]$ is $\max(4,1+1) = 4$-balanced. ◇

Now we can do t cascades of $\overrightarrow{\zeta}^3$ to obtain the cascade $\overrightarrow{\zeta}^4[p,t] = \sum_{i=0}^{t-1} \overrightarrow{\zeta}^3[p-i,t-2]$.

▶ Claim 27. For $3 \leq t \leq p+2/3$ the coordinates of $\overrightarrow{\zeta}^4[p,t] = \sum_{i=0}^{t-1} \overrightarrow{\zeta}^3[p-i,t-2]$ satisfy: For $p \geq j \geq p-t+2$, $\zeta_j^4 = 1$; for $p-t \geq j \geq p-2t+2$, $\zeta_j^4 = -2$; for $p-2t \geq j \geq p-3t+2$, $\zeta_j^4 = 1$; and $\zeta_j^4 = 0$ for all the others j. Furthermore it is 6-balanced.

Proof. This follows easily from the values of the coordinates of $\overrightarrow{\zeta}^3[p-i,t-2]$. Note that $\zeta_{p-t+1}^4 = -1+1 = 0$ and $\zeta_{p-2t+1}^4 = 1-1 = 0$.

We will prove by induction that $\sum_{i=0}^{r} \overrightarrow{\zeta}^3[p-i,t-2]$ is 6-balanced for any $r \leq t-1$. That is true for $r = 0$ as $\overrightarrow{\zeta}^3[p,t-2]$ is 4-balanced. Suppose that $\sum_{i=0}^{r} \overrightarrow{\zeta}^3[p-i,t-2]$ is 6-balanced for some $r \leq t-2$. We apply Lemma 17 with $\overrightarrow{\Phi}^1 = \sum_{i=0}^{r} \overrightarrow{\zeta}^3[p-i,t-2]$ and $\overrightarrow{\Phi}^2 = \overrightarrow{\zeta}^3[p-r-1,t-2]$. We have $h_1 = 6$ by induction hypothesis and furthermore all the coefficients of $\overrightarrow{\Phi}^1$ are greater than -2 by Claim 26 when $r = 0$ or Claim 27 when $r > 0$ and so $\min_i \Phi_i^1 = -2$. Finally $\overrightarrow{\Phi}^2$ is 4-balanced and so $\sum_{i=0}^{r+1} \overrightarrow{\zeta}^3[p-i,t-2]$ is also $\max(6,4+2) = 6$-balanced and by induction $\overrightarrow{\zeta}^4[p,t]$ is 6-balanced. ◇

End of the proof of Theorem 13: The cascade $\overrightarrow{\zeta}^4[p,t]$ consists of t cascades $\overrightarrow{\zeta}^3[p-i,t-2]$ each of them consisting of a cascade $\overrightarrow{\gamma}^3[p-i,t-2]$ and $t-1$ 1-deviations. $\overrightarrow{\gamma}^3[p-i,t-2]$itself consists of $t-2$ cascades $\overrightarrow{\gamma}^2[p-i,t-1]$ each of them consisting of $t-1$ cascades $\overrightarrow{\gamma}^1[p-i,t-1]$ each of them consisting of t 3-deviations $\overrightarrow{\gamma}[p-i]$. So the cascade $\overrightarrow{\zeta}^4[p,t]$ contains $t^2(t-1)(t-2) = \theta(t^4)$ 3-deviations (plus $t(t-1)$ 1-deviations which is negligible).

By Claim 27, it is 6-balanced and so if we choose as starting partition one with 6 groups of each size $i, 1 \leq i \leq p-1$ the cascade is valid. It is easy to obtain such a starting partition from the initial partition which consists of n groups of size 1. Indeed we can create a group of any size i; for that we choose a specific group of size 1 and sucessively move with $(i-1)$ 1-deviations one element of $i-1$ other groups of size 1 to form a group of size i. We do it 6 times for each size $i, 1 \leq i \leq p-1$. Of course that is possible only if $n \geq 6p(p-1)/2$.

Finally, note that in order that the coordinates of all cascades have a meaning we must choose p and t such that $p - 3t \geq 0$. Let us choose $p = 3t$; then the number of vertices is $n = 6\sum_{i=0}^{p-1} i = 6p(p-1)/2 = 9t(3t-1) = \theta(t^2)$.

In summary we have built a cascade which contains $\theta(t^4) = \theta(n^2)$ 3-deviations and so $L(3, n) = \Omega(n^2)$.